Journal of Statistical Physics, Vol. 126, No. 6, March 2007 (© 2007)
DOI: 10.1007/s10955-006-9085-z

Weak Coupling and Continuous Limits for Repeated Quantum Interactions

Stéphane Attal[1] and Alain Joye[2]

Received September 20, 2005; accepted March 7, 2006
Published Online: January 30, 2007

We consider a quantum system in contact with a heat bath consisting in an infinite chain of identical sub-systems at thermal equilibrium at inverse temperature β. The time evolution is discrete and such that over each time step of duration τ, the reference system is coupled to one new element of the chain only, by means of an interaction of strength λ. We consider three asymptotic regimes of the parameters λ and τ for which the effective evolution of observables on the small system becomes continuous over suitable macroscopic time scales T and whose generator can be computed: the weak coupling limit regime $\lambda \to 0,\ \tau = 1$, the regime $\tau \to 0,\ \lambda^2\tau \to 0$ and the critical case $\lambda^2\tau = 1,\ \tau \to 0$. The first two regimes are perturbative in nature and the effective generators they determine is such that a non-trivial invariant sub-algebra of observables naturally emerges. The third asymptotic regime goes beyond the perturbative regime and provides an effective dynamics governed by a general Lindblad generator naturally constructed from the interaction Hamiltonian. Conversely, this result shows that one can attach to any Lindblad generator a repeated quantum interactions model whose asymptotic effective evolution is generated by this Lindblad operator.

KEY WORDS: Markovian approximation, repeated quantum interactions

1. INTRODUCTION

This paper is concerned with the study of the weak coupling limit, and variations thereof, of open quantum systems consisting in a small quantum system defined by a Hamiltonian h_0 on a Hilbert space $\mathcal{H}_0$ coupled to an environment or field or

[1] Institut Girard Desargues, Université de Lyon 1, 21 Av. Claude Bernard, 69622 Villeurbanne Cedex, France.
[2] Institut Fourier, Université de Grenoble 1, BP 74, 38402 St.-Martin d'Hères Cedex, France; e-mail: alain.joy@ujf-grenoble.fr

heat bath modelled by an infinite chain of identical independent $n+1$-level sub-systems on $\otimes_{\mathbb{N}^*}\mathcal{H}$, with n finite. The coupling between the distinguished system and the chain is provided by a discrete sequence of interactions of the small system with one individual sub-system of the chain, in the following way: if $\tau > 0$ is a microscopic time scale, over a macroscopic time interval $]0, k\tau]$, $k \in \mathbb{N}^*$, the small system is coupled with elements $1, 2, \ldots, k$ of the chain, in sequence, for the same time τ and with the same interaction of strength λ. The interactions we consider are of the linear minimal coupling type $\sum_{j=0}^{n} V_j^* \otimes a_j + V_j \otimes a_j^*$, where the a_j^*'s and a_j's are creation and annihilation operators relative to the levels of the sub-system and the V_j's are arbitrary operators on $\mathcal{H}_0$. Note that a Hamiltonian formulation of our system necessarily involves a piecewise constant time-dependent generator.

Similar models of repeated quantum interactions are used in physics in quantum optics, in the theory of repeated quantum measurement or in decoherence, see e.g..[4,8,15] Our primary motivation actually comes from the recent paper.[3] It is shown there that such repeated interactions models converge in some subtle limiting procedure to models of open quantum systems in contact with a heat bath consisting in continuous fields of quantum noises, at zero temperature. This result, to which we come back below, justifies in a sense our a priori somewhat non-conventional point of view in considering the chain as a model of heat bath.

The lack of coupling, and thus of coherence, between the elements of the chain allows to expect that an effective continuous dissipative dynamics for pure states or observables on the small system of the form $e^{t\Gamma}$ should emerge when the number k of discrete interactions goes to infinity and the coupling λ with the chain elements is weak, in the familiar weak coupling regime. Recall that this corresponds to choosing $t \in \mathbb{R}^+$ and considering $\mathbb{N} \ni k = t/\lambda^2$ so that the macroscopic time scale equals $T = \tau t/\lambda^2$. When τ is fixed and $\lambda \to 0$, both k and T go to infinity as $1/\lambda^2$. Moreover, in the setting adopted here, we have another parameter at hand which is the microscopic interaction time τ of the small system with each individual element of the chain. It allows us to explore different asymptotic regimes, as τ goes to zero as well, which characterizes the continuous limit, over suitable macroscopic time scales T.

One goal of this paper is to establish the existence of effective continuous Markovian dynamics in weak and/or continuous limits defining three asymptotic regimes of the parameters (λ, τ). We consider successively the effective Schrödinger evolution on the small system at zero temperature and, when the chain is at equilibrium at zero or positive temperature, the Heisenberg evolution of observables on the small system. The analysis relies on the following property of the model, which is inherent to its definition. The effective dynamics on the small system of pure states or of observables from time 0 to time $k\tau$ is shown to be given by the kth power of a linear operator, which depends on the parameters λ and τ. This expresses the Markov property in a discrete setting.

The first part of the paper is devoted to the usual weak limit regime $\lambda \to 0$, τ fixed and $T = t\tau/\lambda^2 \to \infty$, $0 < t$ finite. While the existence of an effective dynamics obtained by a weak limit procedure is proven for a large class of time-independent Hamiltonian systems, as well as in certain time-dependent situations, see e.g.,[6,7,9,10,12] this question is not addressed in the literature for the case under study. We show the existence an effective dynamics driven by a τ dependent generator which we determine. This first result is obtained by adapting the arguments developed in the study of the weak coupling regime for stationary Hamiltonians to our discrete quantum dynamics framework.

The method is then extended to accommodate the whole range $\tau \to 0$, $\lambda^2\tau \to 0$ over macroscopic time scales $T = t/(\tau\lambda^2) \to \infty$, which defines our second regime. This gives rise to an effective dynamics driven by a τ independent generator we compute as well and to which we come back below. The analysis of these first two regimes is strongly related to regular perturbation theory in the parameter $\lambda^2\tau$ and we refer to these regimes as perturbative regimes. Technically, the study of the second regime relies on an asymptotic analysis in the two parameters λ and τ of the discrete evolution of our system. The divergence of the macroscopic time scale imposes, as usual, some renormalization of the dynamics by the restriction of the uncoupled dynamics. Finally, note that in the second regime, the interaction strength λ is not required to go to zero and can even diverge. The common feature of the generators of the dynamics of observables obtained in these first two regimes is that they commute with the generator $i[h_0, \cdot]$ of the uncoupled unitary evolution restricted to $\mathcal{H}_0$. In other words, the corresponding effective dynamics admits the commutant of h_0 as a non trivial invariant sub-algebra of observables. This property is well known in the weak coupling regime for time-independent Hamiltonians.[7,9,12]

To motivate the study of our third regime, let us come back to the use of repeated quantum interactions models in Ref. 3 which allows to derive effective open quantum systems in some asymptotic regime. The limiting procedure of Ref. 3 gives rise in a natural and spontaneous way to effective dynamics on the Hilbert space $\mathcal{H}_0$ of the small system governed by quantum Langevin equations. The techniques used are those of Quantum Stochastic Calculus; see the volumes[2] for an introduction. The considered limit involves at the same time the time scale τ, the strength of interaction λ as well as a notion of spacing between the subsystems forming the chain in an intricate way. While reminiscent of weak coupling methods in spirit, the limiting procedure of Ref. 3 is nevertheless distinct from the weak coupling limit. Indeed, while $\tau \to 0$, the product $\lambda^2\tau$ is kept constant in Ref. 3, which leads us beyond the perturbative regime. Hence the definition of a third regime given by the critical scaling $\lambda^2\tau = 1$ and $\tau \to 0$ in our repeated quantum interactions model. The relevant macroscopic time scale in this regime is $T = t/(\lambda^2\tau) = t$, which is finite. The goal is to derive an effective evolution of observables for a chain at inverse temperature β which we compare with the results of Ref. 3.

With this scaling, we show that an effective Heisenberg dynamics for observables on $\mathcal{H}_0$ emerges at any temperature. It is generated by a general Lindblad operator whose dissipative part is explicitly constructed in terms of the V_j's defining the coupling in the Hamiltonian, whereas its conservative part is simply $i[h_0, \cdot]$. At zero temperature, we recover the effective Heisenberg dynamics of observables on $\mathcal{H}_0$ of Ref. 3 obtained by means of quantum noises. At positive temperature, our generator coincides with a construction proposed in Ref. 13 for certain models using an a priori modelization of the heat bath by some thermal quantum noises, generalizing those used at zero temperature. For any temperature, the effective dynamics is distinct from that obtained in the previous two perturbative regimes. In particular, the generator obtained does not commute with $i[h_0, \cdot]$ anymore; the generator of the uncoupled evolution restricted to $\mathcal{H}_0$. Hence, there is no obvious sub-algebra of observables left invariant by the effective dynamics of observables. The analysis of this critical case makes use of Chernoff's Theorem, rather than perturbative methods.

Let us compare the generator of the effective dynamics of observables obtained in the regime $\tau \to 0, \tau\lambda^2 \to 0$, and the general Lindblad operator obtained as $\tau \to 0$ with $\tau\lambda^2 = 1$. In the former case, the generator is obtained from the dissipative part of the Lindblad operator of the latter case by retaining its diagonal terms only with respect to the spectral decomposition of the uncoupled evolution restricted to $\mathcal{H}_0$. Or, in an equivalent way, by performing a time average of the Lindblad operator with respect to the uncoupled evolution restricted to $\mathcal{H}_0$. This defines the so-called # operation that makes the commutant of h_0 invariant under the effective dynamics in the regime $\tau \to 0, \tau\lambda^2 \to 0$ (and in the weak coupling regime as well). Our results show that the # operation is present as long as $\lambda^2\tau \to 0$, whereas it disappears in the critical regime $\tau\lambda^2 = 1$. In other words, in the regime $\tau \to 0, \tau\lambda^2 \to 0$, a non-trivial distinguished invariant sub-algebra of observables exists, whereas in the critical case $\tau \to 0, \tau\lambda^2 = 1$, there is a priori no sub-algebra left invariant by the effective dynamics, since its generator takes the form of a generic Lindblad operator.

We finally note here that from a practical point of view, the modelization of the dynamics of observables (or states) of a small system in contact with a reservoir at a certain temperature often starts with a choice of a certain Lindblad generator suited to the physical phenomena to be discussed. Our analysis allows to assign to any Lindblad generator a simple model of repeated quantum interactions, with explicit couplings constructed from the Lindblad generator, whose effective dynamics in the limit $\tau \to 0, \lambda = 1/\sqrt{\tau}$, is generated by the chosen Lindblad operator.

The paper is organized as follows. The general setup and definition of the model are provided in the next section. For the reader's convenience we also state there our main result in the Heisenberg picture for positive temperature in a an abridged form. Section 3 is devoted to the analysis of the weak limit of the model at zero temperature, in the Schrödinger picture. Our main results in this setup are expressed as Corollary 3.2 for the weak coupling regime and Corollary 3.3 for the regime $\lambda^2\tau \to 0$, $\tau \to 0$. This section also contains the technical basis underlying our perturbative analysese in both the Schrödinger and Heisenberg pictures. The main technical result, of independent interest, is actually valid in a Banach space framework and is stated as Theorem 3.1. The full blown positive temperature case, in the Heisenberg picture is dealt with in Sec. 4. The generators of the effective dynamics of observables in the two perturbative regimes are given in Theorem 4.1 and Corollary 4.1. The analysis of the critical regime $\lambda^2\tau = 1$ is presented in Sec. 5, for both the Schrödinger and Heisenberg pictures. Section 6 is devoted to a thorough analysis of the first non-trivial case where both the small system and the elements of the chain consist in two-level systems.

2. REPEATED INTERACTION MODEL AND TYPICAL RESULT

2.1. The Model

Consider the following setup to start with. Our small system, described by the Hilbert space $\mathcal{H}_0$ of dimension $d+1>1$ and a self-adjoint Hamiltonian h_0, interacts with an infinite chain of identical finite dimensional sub-systems modelling a field or heat bath, by means of a time dependent Hamiltonian. The total Hilbert space is $\mathcal{H}_0 \otimes \mathcal{H}$, where $\mathcal{H} = \otimes_{j\geq 1}\mathbb{C}^{n+1}$, $n \geq 1$, is defined in the usual way, see 2.7.2 in Ref. 5. We will call the jth Hilbert space $\mathbb{C}_j^{n+1} \equiv \mathbb{C}^{n+1}$, the Hilbert space at site j, $j = 1, 2, \ldots$ and, following the usage when $n = 1$, we will call the subsystem at site j the spin at site j. We adopt the following convenient notations used in Ref. 3. The vacuum $\Omega \in \mathcal{H}$ is defined as the infinite tensor product of the vacuum vector $\omega = (0 \ \cdots \ 0\ 1)^T$ in $\mathbb{C}^{n+1}$,

$$\Omega = \omega \otimes \omega \otimes \omega \otimes \cdots \in \ \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \otimes \cdots. \tag{2.1}$$

Denoting the ith excited vectors $x_i = (0 \ \cdots \ 0\ 1\ 0 \ \cdots \ 0)^T$, where the 1 sits at the ith line, starting from the bottom, $i = 1, 2, \ldots, d$, the corresponding excited state at site $j \geq 1$ is given by

$$x_i(j) = \omega \otimes \cdots \otimes \omega \otimes x_i \otimes \omega \otimes \cdots, \tag{2.2}$$

where x_i sits at site $j \geq 1$. More generally, given a finite set

$$S = \{(k_1, i_1), (k_2, i_2), \ldots, (k_m, i_m)\} \subset (\mathbb{N}^* \times \{1, 2, \ldots, d\})^m \text{ with all } k_j\text{'s distinct,} \tag{2.3}$$

we define X_S as the vector given by an infinite tensor product as above, with i_jth excited vectors $x_{i_j}(k_j)$ at all sites $k_j \geq 1$, $j = 1, \ldots, m$, and ground state vectors ω everywhere else. $\mathcal{H}$ is the completion under the norm arising from the inner product of linear combinations of such vectors. This construction together with the vacuum $\Omega \equiv X_\emptyset$ yield an orthonormal basis of $\mathcal{H}$, when S runs over all finite sets of the type above.

Let us introduce creation and annihilation operators associated with the vectors $x_i(j)$. Let a_i and a_i^*, $i = 1, 2, \ldots, n$, denote the operators corresponding to $\{\omega, x_1, \ldots, x_n\}$ in $\mathbb{C}^{n+1}$, i.e. such that

$$\begin{array}{lll} a_i x_i = \omega, & a_i \omega = a_i x_j = 0, & \text{if } j \neq i, \\ a_i^* \omega = x_i, & a_i^* x_j = 0 & \text{for any } j = 1, 2, \ldots, n. \end{array} \tag{2.4}$$

Note that these operators do not coincide with the familiar creation and annihilation, however, for i fixed, they satisfy the anti-commutation rules when restricted to the two dimensional subspace $\langle \omega, x_i \rangle$ and are zero on the orthogonal complement of this subspace. Then, for $j \geq 1$, the operators $a_i(j)$ and $a_i(j)^*$ on $\mathcal{H}$ are defined as acting as a_i and a_i^* on the jth copy of $\mathbb{C}^{n+1}$ at site j, and as the identity everywhere else. Therefore, when acting on different copies of $\mathbb{C}^{n+1}$, these operators commute. In keeping with the notations for the reservoir, we introduce a basis of eigenvectors of h_0 for $\mathcal{H}_0$ of the form

$$\{\omega, x_1, x_2, \ldots, x_d\}, \quad \text{where} \quad d = \dim(\mathcal{H}_0) - 1. \tag{2.5}$$

Note that $d \neq n$ in general, but we shall nevertheless use sometimes the notation $\omega(0)$ and $\{x_i(0)\}_{i=1,2,\ldots,d}$ to denote these vectors. No confusion should arise with vectors of $\mathcal{H}$ above, since we labelled the sites of the spins by positive integers. In some cases, $\mathcal{H}_0$ will be an infinite dimensional separable Hilbert space, which corresponds formally to $d = \infty$.

Our formal time dependent Hamiltonian $H(t, \lambda)$ on $\mathcal{H}_0 \otimes \mathcal{H}$ has the form

$$H(t, \lambda) = H_0 + H_F + \lambda H_I(t), \tag{2.6}$$

where

$$H_0 = h_0 \otimes \mathbb{I}, \quad H_F = \sum_{j=1}^{\infty} \sum_{i=1}^{n} \mathbb{I} \otimes \delta_i a_i(j)^* a_i(j), \quad \text{with} \quad \delta_i \in \mathbb{R}, \tag{2.7}$$

and, for $t \in [\tau(k-1), \tau k[$,

$$H_I(t) = \sum_{i=1}^{n} V_i^* \otimes a_i(k) + V_i \otimes a_i(k)^* \equiv I(k), \tag{2.8}$$

where the V_i's and h_0 are bounded operators on $\mathcal{H}_0$, in case $\mathcal{H}_0$ is a separable infinite dimensional Hilbert space. These operators describe the interaction between the small system with the different levels of energy δ_i of the spin at site k, during the time interval $[\tau(k-1), \tau k]$ of length τ. The form of H_F makes it an unbounded operator, but, as we will see in the sequel, we will only make use of the unitary evolution it generates and, moreover it will always be sufficient to work with subspaces containing finitely many excited states only.

In order to make the notations more compact, we introduce vectors with operator valued entries that allow to get rid of the indices $i = 1, \dots, n$. Let

$$a(j)^\sharp = (a_1(j)^\sharp \quad a_2(j)^\sharp \quad \cdots \quad a_n(j)^\sharp)^T \tag{2.9}$$

$$V^\sharp = (V_1^\sharp \quad V_2^\sharp \quad \cdots \quad V_n^\sharp) \tag{2.10}$$

where $\sharp$ denotes either nothing or $*$. Then, using the rules of matrix composition, we can write

$$V^{\sharp_1} \otimes a(j)^{\sharp_2} = \sum_{i=1}^{n} V_i^{\sharp_1} \otimes a_i^{\sharp_2}(j), \tag{2.11}$$

so that we can rewrite the interaction Hamiltonian for $t \in]\tau(k-1), \tau k]$ as

$$I(k) = V^* \otimes a(k) + V \otimes a(k)^*. \tag{2.12}$$

Similarly, with

$$a(j)^\sharp a(j) = (a_1(j)^\sharp a_1(j) \quad a_2(j)^\sharp a_2(j) \quad \cdots \quad a_n(j)^\sharp a_n(j))^T \tag{2.13}$$

$$\delta = (\delta_1 \quad \delta_2 \quad \cdots \quad \delta_n). \tag{2.14}$$

we can write

$$H_F = \mathbb{I} \otimes \sum_{j \geq 1} \delta a(j)^* a(j). \tag{2.15}$$

We will denote the corresponding evolution operator between the time $\tau(k-1)$ and τk by U_k, so that

$$U_k = e^{-i\tau(H_0 + H_F + \lambda I(k))}, \tag{2.16}$$

and the evolution from 0 to τn is given by

$$U(n, 0) = U_n U_{n-1} \cdots U_k \cdots U_1. \tag{2.17}$$

Although not explicited in the notation, the operator $U(n, 0)$ depends on λ and τ.

We will first be interested in the weak coupling limit of this evolution operator characterized by the familiar scaling

$$n = t/\lambda^2, \quad \lambda \to 0 \quad \text{and} \quad \tau \quad \text{fixed}. \tag{2.18}$$

Hence, the macroscopic time scale T is given by

$$T = \tau n = \tau t/\lambda^2 \to \infty. \tag{2.19}$$

Note, however, that in contrast with the usual set up, we have here a non-smooth time dependent Hamiltonian $H(t, \lambda)$. Also, the time dependence lies in the coupling term, and not on the small system's Hamiltonian, as considered in Ref. 10

Remarks

(i) In order not to bury the main points of our analysis under technical subtleties, we have chosen to work in a simple framework where all relevant operators are bounded or matrix valued. Nevertheless, some of our results below hold if we consider our heat bath to live in a tensor product of infinite dimensional separable Hilbert spaces and make further assumptions so that the field Hamiltonian and interaction are bounded.
(ii) In some cases we shall allow $\mathcal{H}_0$ to be a separable Hilbert space. This will be explicitly stated in the hypotheses. Otherwise, we will work on the model defined above, under the general assumption

HO: The Hamiltonian is defined on the Hilbert space $\mathcal{H}_0 \otimes \mathcal{H}$, where $\mathcal{H}_0 = \mathbb{C}^{d+1}, \mathcal{H} = \otimes_{j\geq 1}\mathbb{C}^{n+1}$, for d, n finite, and is given by (2.6)–(2.8). The evolution it generates is given by (2.17) and (2.16).

2.2. Typical Positive Temperature Results

Before turning to the mathematical analysis, we want to provide the reader with the flavour of the results to come, by giving here some informal statements about the Heisenberg evolution of observables at positive temperature. We refer the reader to Sec. 4 for more details and for the full blown corresponding Theorems.

Consider the equilibrium state $\omega(\beta)_N$ of a chain of N spins at inverse temperature β of the form $\omega(\beta)_N = r(\beta) \otimes r(\beta) \otimes \cdots \otimes r(\beta)$, where $r(\beta) = e^{-\beta\delta a^* a}/\mathcal{Z}(\beta)$, and $\mathcal{Z}(\beta)$ is the partition function ensuring normalization. The Heisenberg evolution of an observable $B \in \mathcal{L}(\mathcal{H}_0)$ on the small system is defined by

$$B_\beta(k, \lambda, \tau) = \mathrm{Tr}_{\mathcal{H}}((\mathbb{I} \otimes \omega_N(\beta))\, U(k, 0)^{-1}(B \otimes \mathbb{I}_{\mathcal{H}})U(k, 0)) \in \mathcal{L}(\mathcal{H}_0), \quad (2.20)$$

where $\mathrm{Tr}_{\mathcal{H}}(A)$ denotes the partial trace of the observable $A \in \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H})$ taken on the spin variables only. See Sec. 4. Let $\mathcal{U}_{0,0}(0)$ be the free evolution acting on $\mathcal{L}(\mathcal{H}_0)$ as $\mathcal{U}_{0,0}(0)(B) = e^{i\tau h_0} B e^{-i\tau h_0}$, and denote by $\{P_l\}_{l=1,\ldots,r}$ the set of spectral projectors of $\mathcal{U}_{0,0}(0)$, viewed as a linear opeator on $\mathcal{L}(\mathcal{H}_0)$. The sharp operation on $K \in \mathcal{L}(\mathcal{H}_0)$ is then defined by $K^\# = \sum_{l=1}^r P_l K P_l$. Let t be any positive fixed number.

Our first result in this setup is the following weak limit statement for $k = t/\lambda^2$: there exists T_β acting on $\mathcal{L}(\mathcal{H}_0)$, such that

$$\lim_{\substack{\lambda\to 0\\ t/\lambda^2\in\mathbb{N}}} \mathcal{U}_{0,0}(0)^{-t/\lambda^2} B_\beta(t/\lambda^2, \lambda, \tau) = e^{t\Gamma_\beta^w}(B),$$

were $\Gamma^w_\beta(B) = \frac{1}{\mathcal{Z}(\beta)}(\mathcal{U}_{0,0}(0)^{-1}T_\beta)^\#(B)$. The operator Γ^w_β actually depends on τ and possesses a complicated, though explicit form, displayed in Lemma 4.6.

Allowing τ to go to zero and considering the scaling $k = t(\tau\lambda)^2$, we get

$$\lim_{\substack{\tau\to 0, \lambda^2\tau\to 0\\ t/(\tau\lambda)^2\in\mathbb{N}}} \mathcal{U}_{0,0}(0)^{-t/(\tau\lambda)^2} B_\beta(t/(\tau\lambda^2), \lambda, \tau) = e^{t\Gamma^\#_\beta(B)},$$

were Γ_β is the dissipative part of a β-dependent Lindblad operator constructed by means of the V_m's. It is rather simple and given explicitely in (4.62).

Our last result concerns the critical regime $\tau\lambda^2 = 1$ and reads

$$\lim_{\substack{\tau\to 0\\ t/\tau\in\mathbb{N}}} B_\beta(t/\tau, 1/\sqrt{\tau}, \tau) = e^{t(i[h_0,\cdot]+\Gamma_\beta(\cdot))}(B) \tag{2.21}$$

with a full Lindblad generator $i[h_0, \cdot] + \Gamma_\beta(\cdot)$, whose dissipative part Γ_β coincides with the one of the previous result.

3. WEAK LIMIT OF THE SCHRÖDINGER REPRESENTATION AT ZERO TEMPERATURE

As a warm up, and in order to derive some preliminary estimates, we prove here the existence of the weak limit for our model at zero temperature in the Schrödinger picture, and compute this limit. We first prove a key lemma that reduces the computation of the projected part of the evolution $U(n, 0)$ (2.17) to nth power of a single matrix. Then we perform a general analysis of large powers of operators based on perturbative expansions which appear in the computations of weak limits. These technical results are expressed in Proposition 3.2 and Theorem 3.1 under different sets of hypotheses. Their applications to our model are given in Corollaries 3.2 and 3.3.

3.1. Markov Properties

Let P be the projection from $\mathcal{H}_0 \otimes \mathcal{H}$ to the subspace $\mathcal{H}_0 \otimes \mathbb{C}\Omega$ defined by

$$P = \mathbb{I} \otimes |\Omega\rangle\langle\Omega|. \tag{3.1}$$

The object of interest to us in this Section will thus be the limit

$$\lim_{\lambda\to 0} P(U(t/\lambda^2, 0))P, \tag{3.2}$$

as an operator from $P\ \mathcal{H}_0 \otimes \mathcal{H}$ to $P\ \mathcal{H}_0 \otimes \mathcal{H}$, identified with $\mathcal{H}_0$, the Hilbert space of the small system.

Note that

$$U_j = e^{-i\tau\widehat{H}_j}e^{-i\tau\widetilde{H}_j}, \tag{3.3}$$

with

$$\begin{aligned}\widetilde{H}_j &= h_0 \otimes \mathbb{I} + \mathbb{I} \otimes \delta a(j)^* a(j) + \lambda(V^* \otimes a(j) + V \otimes a(j)^*)\\ \widehat{H}_j &= \mathbb{I} \otimes \sum_{k \neq j} \delta a(k)^* a(k),\end{aligned} \tag{3.4}$$

two operators that commute.

We observe the following property of products of operators U_k, which shows the Markovian nature of the reduced evolution.

Lemma 3.1. *Let us write the restriction of U_j to $\mathcal{H}_0 \otimes \mathbb{C}_j^{n+1}$ as a block matrix with respect to the ordered basis of $\mathcal{H}_0 \otimes \mathbb{C}_j^{n+1}$.*

$$\begin{gathered}\{\omega \otimes \omega,\ x_1 \otimes \omega, \ldots, x_d \otimes \omega,\\ \omega \otimes x_1, x_1 \otimes x_1, \ldots, x_d \otimes x_1,\\ \vdots\\ \omega \otimes x_n, x_1 \otimes x_n, \ldots, x_d \otimes x_n\}\end{gathered} \tag{3.5}$$

as

$$U_j|_{\mathcal{H}_0 \otimes \mathbb{C}_j^{n+1}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.6}$$

where A is a $(d+1) \times (d+1)$ matrix, B is $(d+1) \times n(d+1)$, C is $(d+1)n \times (d+1)$ and D is $n(d+1) \times n(d+1)$. Then, for any $m \geq 0$,

$$PU(m, 0)P = A^m \otimes |\Omega\rangle\langle\Omega| \simeq A^m. \tag{3.7}$$

Proof Follows from the fact that

$$U_j(\mathbb{I} \otimes |\Omega\rangle\langle\Omega|) = e^{-i\tau \widetilde{H}_j}(\mathbb{I} \otimes |\Omega\rangle\langle\Omega|) \tag{3.8}$$

where, if $\mathcal{H}_0 \ni v = v_0\omega(0) + \sum_{i=1}^d v_i x_i(0) \simeq \vec{v}$,

$$e^{-i\tau \widetilde{H}_j} v \otimes \Omega = A\vec{v} \otimes \Omega + \sum_{i=1}^{n+1} (C\vec{v})_i \otimes x_i(j), \tag{3.9}$$

where $(\vec{w})_i$ denotes the i'th component of the vector $\vec{w}$. Hence, due to the fact that different U_j's act on different $\mathbb{C}_j^{n+1}$'s

$$U_m U_{m-1} \cdots U_1 v \otimes \Omega = A^m \vec{v} \otimes \Omega + \sum_{i=1}^{m(d+1)} \vec{w}_i \otimes X_{S_i}, \tag{3.10}$$

where $\vec{w}_i$ are some vectors in $\mathbb{C}^{d+1}$ and the excited sets S_i are never empty. Therefore their contribution vanishes in the computation

$$(\mathbb{I} \otimes |\Omega\rangle\langle\Omega|) U_m U_{m-1} \cdots U_1 v \otimes \Omega = A^m \vec{v} \otimes |\Omega\rangle\langle\Omega|. \tag{3.11}$$

□

As is easy to check along the same lines, in case $\mathcal{H}_0$ is infinite dimensional, we can generalize the above Lemma as follows.

Lemma 3.2. *Let $\mathcal{H}_0$ be a separable Hilbert space and $h_0, V_j, j = 1, \ldots, n$ be bounded on $\mathcal{H}_0$. We set*

$$P_j = |x_j\rangle\langle x_j| : \mathbb{C}^{n+1} \mapsto \mathbb{C}^{n+1},\ j = 1, \ldots, n,\ P_0 = |\omega\rangle\langle\omega| \quad and \quad Q_0 = \mathbb{I} - P_0, \tag{3.12}$$

so, that

$$\mathcal{H}_0 \otimes \mathbb{C}^{n+1} = (\mathcal{H}_0 \otimes P_0\mathbb{C}^{n+1}) \oplus (\mathcal{H}_0 \otimes Q_0\mathbb{C}^{n+1}) \simeq (\mathcal{H}_0 \otimes \mathbb{C}) \oplus (\mathcal{H}_0 \otimes \mathbb{C}^n). \tag{3.13}$$

We can decompose

$$U_j|_{\mathcal{H}_0 \otimes \mathbb{C}_j^{n+1}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.14}$$

where $A : \mathcal{H}_0 \mapsto \mathcal{H}_0$, $B : \mathcal{H}_0 \otimes \mathbb{C}^n \mapsto \mathcal{H}_0$, $C : \mathcal{H}_0 \mapsto \mathcal{H}_0 \otimes \mathbb{C}^n$ and $D : \mathcal{H}_0 \otimes \mathbb{C}^n \mapsto \mathcal{H}_0 \otimes \mathbb{C}^n$. Then, for any $m \geq 1$,

$$PU(m, 0)P = A^m \otimes |\Omega\rangle\langle\Omega| \simeq A^m. \tag{3.15}$$

The above Lemmas thus lead us to consider a reduced problem on $\mathcal{H}_0$. We need to compute the matrix A in the decomposition (3.6) of $e^{-i\tau(h_0+\delta a^*a+\lambda(V^*a+Va^*))}$, where we dropped the indices j, the $\mathbb{I}$ and the $\otimes$ symbol in the notation. Recall however that a summation over the excited states of H_F is implicit in the notation.

3.2. Preliminary Estimates

In order to apply perturbation theory as $\lambda \to 0$ and, later on, in other regimes involving $\tau \to 0$ as well, we derive below estimates to be used throughout the paper.

We rewrite the generator as

$$H(\lambda) = H(0) + \lambda W, \quad \text{with} \quad H(0) = h_0 + \delta a^*a \quad \text{and} \quad W = V^*a + Va^*. \tag{3.16}$$

With a slight abuse of notations, the projector P takes the form

$$P = \mathbb{I} - a^*a \tag{3.17}$$

We can slightly generalize the setup and work under the following hypothesis:

> **H1:** Let P be a projector on a Banach space $\mathcal{B}$ and $H(\lambda)$ be an operator of the form
> $$H(\lambda) = H(0) + \lambda W, \tag{3.18}$$
> where $H(0)$ and W are bounded and $0 \leq \lambda \leq \lambda_0$ for some $\lambda_0 > 0$. Further assume that
> $$[P, H(0)] = 0 \quad \text{and} \quad W = PWQ + QWP \quad \text{where} \quad Q = \mathbb{I} - P. \tag{3.19}$$

We consider

$$U_\tau(\lambda) = e^{-i\tau H(\lambda)}. \tag{3.20}$$

For later purposes, we also take care of the dependence in τ of the error terms. As this parameter will eventually tend to zero in some applications to come below, we consider the error terms as both λ and τ tend to zero, independently of each other. We have a first easy perturbative result

Lemma 3.3. *Let **H1** be true. Then, as λ and τ go to zero,*

$$e^{-i\tau(H(0)+\lambda W)} = e^{-i\tau H(0)} + \lambda F(\tau) + \lambda^2 G(\tau) + O(\lambda^3\tau^3) \tag{3.21}$$

$$Pe^{-i\tau(H(0)+\lambda W)}P = Pe^{-i\tau H(0)}P + \lambda^2 PG(\tau)P + PO(\lambda^4\tau^4)P, \tag{3.22}$$

where

$$\begin{aligned} F(\tau) &= \sum_{n\geq 1} \frac{(-i\tau)^n}{n!} \sum_{\substack{m_j\in\mathbb{N} \\ m_1+m_2=n-1}} H(0)^{m_1} W H(0)^{m_2} \\ &= -ie^{-i\tau H(0)} \int_0^\tau ds_1 e^{is_1 H(0)} W e^{-is_1 H(0)} \end{aligned} \tag{3.23}$$

$$\begin{aligned} G(\tau) &= \sum_{n\geq 2} \frac{(-i\tau)^n}{n!} \sum_{\substack{m_j\in\mathbb{N} \\ m_1+m_2+m_3=n-2}} H(0)^{m_1} W H(0)^{m_2} W H(0)^{m_3} \\ &= -e^{-i\tau H(0)} \int_0^\tau ds_1 \int_0^{s_1} ds_2 e^{-is_1 H(0)} W e^{-i(s_1-s_2)H(0)} W e^{-is_2 H(0)} \end{aligned} \tag{3.24}$$

Moreover

$$\frac{\mathrm{d}}{\mathrm{d}\tau} G(\tau) = -iH(0)G(\tau) - iWF(\tau), \; G(0) = 0 \tag{3.25}$$

$$F(-\tau) = -e^{i\tau H(0)} F(\tau) e^{i\tau H(0)} \tag{3.26}$$

$$G(-\tau) = -e^{i\tau H(0)} G(\tau) e^{i\tau H(0)} + e^{i\tau H(0)} F(\tau) e^{i\tau H(0)} F(\tau) e^{i\tau H(0)}. \tag{3.27}$$

Remark. *Formula (3.21) is true without assuming that* W *is off-diagonal with respect to* P *and* Q.

Proof First note that $U_\tau(\lambda) = e^{-i\tau H(\lambda)}$ is analytic in both variables λ and τ in $\mathbb{C}^2$. Then, we compute the exponential of $-i\tau$ times $H(\lambda)$ as a convergent series. Consider terms of the form

$$\begin{aligned}(H(0)+\lambda W)^n &= H(0)^n + \lambda \sum_{k=0}^{n-1} H(0)^k W H(0)^{n-1-k} \\ &\quad + \lambda^2 \sum_{\substack{m_j \in \mathbb{N} \\ m_1+m_2+m_3=n-2}} H(0)^{m_1} W H(0)^{m_2} W H(0)^{m_3} + O(\lambda^3 C^n).\end{aligned} \tag{3.28}$$

The error term in C^n comes from the boundedness of the operators involved. Multiplication by $(-i\tau)^n/n!$ and summation over $n \geq 0$ yields the first result with our definition of $F(\tau)$ and $G(\tau)$. The second result follows from taking into account that $PWP = QWQ = 0$, hence only the terms with an even number of W's survive and we get

$$P\frac{(H(0)+\lambda W)^n}{n!}P = \tag{3.29}$$

$$P\left(\frac{H(0)^n}{n!} + \lambda^2 \sum_{\substack{m_j \in \mathbb{N} \\ m_1+m_2+m_3=n-2}} \frac{H(0)^{m_1} W H(0)^{m_2} W H(0)^{m_3}}{n!} + O(\lambda^4 C^n/n!)\right)P.$$

The overall error in $\tau^4\lambda^4$ comes from the fact that it takes at least four terms in (3.28) to get a contribution of order λ^4. The computation above was conducted to order λ^2 because of the scaling (2.18). The order λ term $F(\tau)$ doesn't contribute, being off diagonal with respect to P.

An alternative derivation of a perturbation series of $e^{-i\tau(H(0)+\lambda W)}$ in λ yields the other expressions for $F(\tau)$ and $G(\tau)$. It is obtained via Dyson series in the familiar interaction picture. We have the identity

$$i\frac{d}{d\tau}e^{-i\tau(H(0)+\lambda W)} = (H(0)+\lambda W)e^{-i\tau(H(0)+\lambda W)}, \; e^{-i\tau(H(0)+\lambda W)}\Big|_{\tau=0} = \mathbb{I}. \tag{3.30}$$

Introducing

$$\Theta(\lambda, \tau) = e^{i\tau H(0)} e^{-i\tau(H(0)+\lambda W)}, \tag{3.31}$$

this operator satisfies

$$i \frac{d}{d\tau} \Theta(\lambda, \tau) = \lambda e^{i\tau H(0)} W e^{-i\tau H(0)} \Theta(\lambda, \tau), \quad \Theta(\lambda, \tau)|_{\tau=0} = \mathbb{I}. \tag{3.32}$$

Hence we have the convergent expansion

$$\Theta(\lambda, \tau) = \sum_{n=0}^{\infty} (-i\lambda)^n \int_0^{\tau} ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n e^{is_1 H(0)} W$$
$$\times e^{-i(s_1-s_2)H(0)} W e^{-i(s_2-s_3)H(0)} \cdots e^{-i(s_{n-1}-s_n)H(0)} W e^{-is_n H(0)}. \tag{3.33}$$

Therefore, focusing on the terms of order λ and λ^2, we get the alternative expressions for $F(\tau)$ and $G(\tau)$.

The differential equation yielding $G(\tau)$ as a function of $F(\tau)$ follows from explicit computations on the expressions above, as the identities for $\tau \mapsto -\tau$. □

Let us give some more properties of the expansion of $U_\tau(\lambda)$ for $\lambda > 0$ small, $\tau > 0$ in the Hilbert space context that will be used later on.

Corollary 3.1. *Assume $\mathcal{B}$ is a Hilbert space, $H(0)$, W and P are self-adjoint and λ, τ are real. As $\lambda \to 0$, the operator $U_\tau(\lambda) = e^{-i\tau H(\lambda)}$ satisfies*

$$U_\tau(\lambda) = e^{-i\tau H(0)} + \lambda F(\tau) + \lambda^2 G(\tau) + O(\lambda^3 \tau^3) \tag{3.34}$$

$$U_\tau(\lambda)^{-1} = U_\tau(\lambda)^* = U_{-\tau}(\lambda)$$
$$= e^{i\tau H(0)} + \lambda F(-\tau) + \lambda^2 G(-\tau) + O(\lambda^3 \tau^3) \tag{3.35}$$

with the identities for all $\tau \in \mathbb{R}$.

$$F(-\tau) = F^*(\tau) \tag{3.36}$$

$$G(-\tau) = G^*(\tau). \tag{3.37}$$

Proof Follows from the fact that $H(\lambda)$ is self-adjoint. □

3.3. Weak Limit Results

The technical basis underlying all our weak limit results is contained in the next two Lemmas and the Proposition following them. They are stated in a general framework that will suit both our analysese of the Schrödinger and Heisenberg representations. This is why we use independ notations.

Lemma 3.4. *Let $V(x), x \in [0, x_0)$, and R be bounded linear operators on a Banach space $\mathcal{B}$ such that, in the operator norm, $V(x) = V(0) + xR + O(x^2)$, and $V(0)$ is an isometry which admits the following spectral decomposition*

$$V(0) = \sum_{j=0}^{r} e^{-iE_j} P_j \quad where \quad r < \infty,\ E_j \in \mathbb{R},\ \{e^{-iE_j}\}_{j=0,\dots,r}\ distinct. \tag{3.38}$$

Let $h = \sum_{j=0}^{r} E_j P_j$ so that $V(0) = e^{-ih}$ and

$$J = \sum_{j,k=0}^{r} \alpha_{jk} P_j R P_k \quad where \quad \alpha_{jk} = \begin{cases} \frac{E_j - E - k}{e^{-iE_j} - e^{-iE_k}} & if\ j \neq k \\ ie^{iE_j} & if\ j = k. \end{cases} \tag{3.39}$$

Then, for any $0 \le t \le t_0$, where t_0 finite, and $t/x \in \mathbb{N}$,

$$\left\| V(x)^{\frac{t}{x}} - e^{-i(h+xJ)^{\frac{t}{x}}} \right\| = O(x), \quad as \quad x \to 0, \quad s.t. \quad t/x \in \mathbb{N}. \tag{3.40}$$

Remarks

(i) Expressing the projectors P_j by Von Neumann's ergodic theorem as

$$P_j = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (e^{iE_j} V(0))^n \tag{3.41}$$

shows that they are of norm one.

(ii) The operator $J = J(R, h)$ is defined to make Eq. (3.43) of order x^2. This equation is a particular case of $i \int_0^1 e^{ish} X e^{-ish} ds = Y$ which is solved in a similar fashion.

Proof With $m = t/x \in \mathbb{N}$,

$$V(x)^m - e^{-i(h+xJ)^m} = \sum_{k=0}^{m-1} V(x)^k \big(V(x) - e^{-i(h+xJ)}\big) e^{-i(h+xtJ)^{m-1-k}} \tag{3.42}$$

where, by hypothesis and Lemma 3.3

$$V(x) - e^{-i(h+xJ)} = x\left(R + ie^{-ih} \int_0^1 e^{ihs} J e^{-ihs} ds\right) + O(x^2). \tag{3.43}$$

Moreover, note also

$$\|V(x)\| = 1 + O(x), \quad \left\|e^{-i(h+xJ)}\right\| = 1 + O(x). \tag{3.44}$$

Our definition (3.39) of J is designed to make the term of order x in (3.43) vanish. Therefore, there exists positive constants c_0, c_1 such that we can estimate for any

$0 \le t \le t_0 < \infty$

$$\left\| V(x)^m - e^{-i(h+xJ)^m} \right\| \le cx^2 \sum_{k=0}^{m-1} \left\| V(x) \right\|^k \left\| e^{-i(h+xJ)} \right\|^{m-1-k}$$

$$\le c_0 x^2 m(1 + c_0 x)^m \le c_0 t x e^{\frac{t}{x} \ln(1+c_0 x)} \le x c_0 t_0 e^{c_1 t_0} = O(x). \quad (3.45)$$

□

It will be necessary to control the dependence of such estimates on a parameter $\tau \to 0$. later on. This will cause no serious difficulty, since all steps are explicited in the argument. To achieve sufficient control in τ, we need to revisit the proof of a well known lemma, which holds under weaker hypothesese than ours, see Davies.[(7)]

Lemma 3.5. *Let $e^{-ih} = \sum_{j=0}^{r} e^{-iE_j} P_j$ be the isometry (3.38) on the Banach space $\mathcal{B}$ and let K be a bounded operator on $\mathcal{B}$. There exists a constant c depending on r and t_0 only, such that for any $t \in [0, t_0]$, t_0 finite,*

$$\left\| e^{ith/x} e^{-i\frac{t}{x}(h+xK)} - e^{-itK^{\#}} \right\| \le c \frac{x\|K\|(1 + \|K\|)e^{2\|K\|t_0}}{\inf_{j\ne k} |E_j - E_k|}, \quad as \quad x \to 0, \tag{3.46}$$

where $K^{\#} = \sum_{j=0}^{r} P_j K P_j = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{ish} K e^{-ish} ds$.

Remark. The expression of $K^{\#}$ as a Cesaro mean is a classical computation which shows that $\|K^{\#}\| \le \|K\|$.

Proof We follow. [(7)] Let $f \in \mathcal{B}$ and

$$f_x(t) = e^{ith/x} e^{-i\frac{t}{x}(h+xK)} f, \quad f(t) = e^{-itK^{\#}} f. \tag{3.47}$$

By the fundamental Theorem of calculus, we can write

$$i(f_x(t) - f(t)) = \int_0^i (e^{ish/x} K e^{-ish/x} f_x(s) - K^{\#} f(s)) ds$$

$$= \int_0^t (e^{ish/x} K e^{-ish/x} (f_x(s) - f(s)) + (e^{ish/x} K e^{-ish/x} - K^{\#}) f(s)) ds. \tag{3.48}$$

Hence,

$$\|f_x(t) - f(t)\| \le \|K\| \int_0^t \|f_x(s) - f(s)\| ds + \mathcal{F}(x, t_0) \tag{3.49}$$

where

$$\mathcal{F}(x, t_0) = \sup_{0 \le t \le t_0} \left\| \int_0^t (e^{ish/x} K e^{-ish/x} - K^{\#}) e^{-isK^{\#}} f \, ds \right\|. \tag{3.50}$$

Now,

$$\begin{aligned} e^{ish/k} K e^{-ish/x} - K^{\#} &= e^{ish/x}(K - K^{\#}) e^{-ish/x} \\ &= \sum_{j \neq k} e^{ish/x} P_j K P_k e^{-ish/x} = \sum_{j \neq k} e^{is(E_j - E_k)/x} P_j K P_k, \end{aligned} \tag{3.51}$$

so that we can integrate (3.50) by parts to obtain

$$\begin{aligned} &\int_0^t (e^{ish/x} K e^{-ish/x} - K^{\#}) e^{-isK^{\#}} f \, ds \\ &= \sum_{j \neq k} \int_0^t \frac{x}{i(E_j - E_k)} \frac{d}{ds} e^{is(E_j - E_k)/x} P_j K P_k e^{-isK^{\#}} f \, ds \\ &= \sum_{j \neq k} \frac{x}{i(E_j - E_k)} e^{is(E_j - E_k)/x} P_j K P_k e^{-isK^{\#}} f \, \Big|_0^t \\ &\quad + \sum_{j \neq k} \int_0^t \frac{x}{(E_j - E_k)} e^{is(E_j - E_k)/x} P_j K P_k K^{\#} e^{-isK^{\#}} f \, ds. \end{aligned} \tag{3.52}$$

Hence, using $\|K^{\#}\| \le \|K\|$, we can bound (3.52) by

$$\sum_{j \neq k} \frac{x \|K\| (2 + t\|K\|) e^{\|K\| t}}{|E_j - E_k|} \|f\|. \tag{3.53}$$

Thus,

$$\mathcal{F}(x, t_0) \le \max(2, t_0)(r^2 - r) \frac{x(1 + \|K\|)\|K\| e^{\|K\| t_0}}{\inf_{j \neq k} |E_j - E_k|}. \tag{3.54}$$

At this point we can invoke Gronwall's Lemma, the above estimate and (3.49) to finish the proof. □

From these two Lemmas, we immediately get the

Proposition 3.1. *Let $V(x)$, $x \in [0, x_0)$ and* R *be bounded operators on a Banach space $\mathcal{B}$ such that, in the operator norm, $V(x) = V(0) + xR + O(x^2)$, where $V(0)$ is an isometry admitting the spectral decomposition $V(0) = \sum_{j=0}^{r} e^{-iE_j} P_j$ and let*

$h = \sum_{j=0}^{r} E_j P_j$. *Then, for any* $0 \le t \le t_0$, *if* $x \to 0$ *in such a way that* $t/x \in \mathbb{N}$,

$$V(0)^{-t/x} V(x)^{t/x} = e^{te^{ih} R^{\#}} + O(x), \quad in\ norm, \tag{3.55}$$

where $K^{\#} = \sum_{j=0}^{r} P_j K P_j$, *for any* $K \in \mathcal{L}(\mathcal{B})$.

Remarks

(i) The operator in the exponent can be rewritten as

$$e^{it} R^{\#} = e^{ih} R^{\#} = (e^{ih} R)^{\#} = (Re^{ih})^{\#}. \tag{3.56}$$

(ii) The hypotheses are made on the isometry $V(0)$, not on the operator h.

We can now derive our first results concerning the weak limit in the Schrödinger picture. We do so in the general setup described in **H1**. We further assume:

H2:
The restriction $H_P(0)$ of $H(0)$ to $P\mathcal{B}$ is diagonalizable and reads

$$H_P(0) = \sum_{j=0}^{r} E_j P_j, \quad \text{with} \quad \dim(P_j) \le \infty, \quad r \text{ finite}. \tag{3.57}$$

Moreover, the operator $Pe^{-i\tau H(0)} = Pe^{-i\tau H_P(0)}$ is an isometry on $P\mathcal{B}$.

Note that this implies $Pe^{-i\tau H_P(0)}$ is invertible and

$$P = \sum_{j=0}^{r} P_j, \quad E_j \in \mathbb{R} \quad \forall j = 0, \dots, r, \quad \text{and} \quad Pe^{-i\tau H(0)} = \sum_{j=0}^{r} e^{-i\tau E_j} P_j, \tag{3.58}$$

where the projectors P_j are eigenprojectors of $Pe^{-i\tau H(0)}$ iff the $e^{-i\tau E_j}$'s are distinct. In case $\mathcal{B}$ is a finite dimensional Hilbert space and $H(0)$ is self adjoint, **H2** is automatically true.

Proposition 3.2. *Let* $H(\lambda)$ *and* P *on* $\mathcal{B}$ *satisfy* ***H1*** *and* ***H2***. *Further assumer* $\tau > 0$ *is such that the values* $\{e^{-i\tau E_j}\}_{j=0}^{r}$ *are distinct. Then, for any* $0 \le t < \infty$,

$$\lim_{\substack{\lambda \to 0 \\ t/\lambda^2 \in \mathbb{N}}} e^{i\tau t H(0)/\lambda^2} \left[Pe^{-i\tau H(\lambda)} P \right]^{t/\lambda^2} = e^{t\Gamma^w(\tau)} \ on \quad P\mathcal{B} \tag{3.59}$$

where

$$\Gamma^w(\tau) = e^{i\tau H(0)} G(\tau)^{\#} = -\int_0^{\tau} ds \int_0^{s} dt\, W e^{-it(H(0)-E_j)} W^{\#}, \tag{3.60}$$

and $K^{\#} = \sum_j^r P_j K P_j$ *for any* $K \in \mathcal{L}(P\mathcal{B})$.

Remarks

(i) In case some values among $\{e^{-i\tau E_j}\}_{j=0}^r$ coincide, the result holds whith the P_j's replaced by Π_j's, the spectral projectors of $Pe^{-i\tau H(0)}|_{P\mathcal{B}}$.

(ii) If, for any $j = 0, \dots, r$, the reduced resolvents $R_Q(E_j) = (H(0) - E_j)\restriction_{Q\mathcal{B}}^{-1}$ all exist, with $Q = \mathbb{I} - P$, then

$$\Gamma^w(\tau) = -\sum_{j=0}^{r} P_j W\, R_Q(E_j)\left(R_Q(E_j) - R_Q(E_j)e^{-i\tau(H(0)-E_j)|_{Q\mathcal{B}}} - i\tau\mathbb{I}\right) W P_j \tag{3.61}$$

(iii) If $\mathcal{B}$ is a Hilbert space, and $H(\lambda)$ is self-adjoint with dim $P_j = 1$, we can express Γ^w in yet another way. We write $P_j = |\varphi_j\rangle\langle\varphi_j|$ and introduce $d\mu_j^W(E)$, $j = 0, \dots, r$, the spectral measures of the vectors $W\varphi_j = QW\varphi_j$, with respect to $H(0)|_{Q\mathcal{B}}$ Then, if $\widehat{\cdot}$ denotes the Fourier transform,

$$\Gamma^w(\tau) = -\sum_{j=0}^{\tau}\int_0^\tau ds\int_0^s dt\ \widehat{\mu_j^W}(t)e^{itEj}P_j. \tag{3.62}$$

Proof of Proposition 3.2 As we are to work in $P\mathcal{B}$, we will write A_P for PAP etc ... Our assumption on τ makes the eigenvalues of $e^{-i\tau H_P(0)}$ distinct so that the P_j's are eigenprojectors of both $H_P(0)$ and $e^{i\tau H_P(0)}$. Then, Lemma 3.3 shows that $V(x) := P(e^{-i\tau(H(0)+\sqrt{x}W)})P$, $x = \lambda^2$, satisfies the hypotheses of Proposition 3.1 with $h = \tau H_P(0)$, $R = G_P(\tau)$ and $\tau > 0$ fixed. Hence the result, making use of $e^{-i\tau H_P(0)} = e^{-i\tau H(0)}P$. The last statement follows from (3.24). □

We are now in a position to state the existence of a contraction semi-group on $P\mathcal{B}$ obtained by means of a weak limit for our specific time dependent Hamiltonian model. The following is a direct application of Proposition 3.2.

Corollary 3.2. *Let U(n,0) be defined on $\mathcal{H}_0 \otimes \mathcal{H}$, where $\mathcal{H}_0$ is separable, by (2.17, 2.16, 2.6), let $P = \mathbb{I} \otimes |\Omega\rangle\langle\Omega|$, and let $\{E_j\}_{j=0,\dots,r}$ be the eigenvalues of h_0 associated with eigenprojectors $\{P_j\}_{j=0,\dots,r}$. Assume the values $\{e^{-\tau E_j}\}_{j=0,\dots,r}$, are distinct. Then, for any fixed $0 \le t < \infty$.*

$$\lim_{\substack{\lambda\to 0\\ t/\lambda^2\in\mathbb{N}}} [e^{i\tau t H(0)/\lambda^2} PU(t/\lambda^2, 0)P] = e^{t\Gamma w(\tau)} \ \ on \ \ P\mathcal{B} \tag{3.63}$$

where

$$\Gamma^w(\tau) = e^{i\tau H(0)}G(\tau)^\# = -\int_0^\tau ds\int_0^s dt\sum_{j=0}^{\tau}\sum_{m-1}^{n} P_j V_m^* e^{-it(h_0+\delta_m-E_j)}V_m P_j \tag{3.64}$$

generates a contraction semi-group and # corresponds to the set of eigenprojectors $\{P_j\}_{j=0,\ldots,r}$.

Remarks

(0) The macroscopic time scale at which we observe the system is $T = \tau t/\lambda^2 \to \infty$.

(i) There are cases where $\Gamma^w(\tau)$ generates a group of isometries.

(ii) Again, if the $e^{-i\tau E_j}$'s are not distinct, we have to take the spectral projectors of $e^{-i\tau h_0}$ instead of the P_j's in the definition of the operation #.

(iii) Note that the effective dynamics commutes with h_0, so that no transition between the eigenspaces of h_0 can take place. However, if the $e^{-i\tau E_j}$'s are not distinct, transitions between different eigenspaces of h_0 corresponding to the same eigenvalue of $e^{-i\tau h_0}$ are possible.

(iv) In case h_0 is non degenerate, $r = d$ and we can write $P_j = |x_j\rangle\langle x_j|$, with x_j the eigenvector associated with E_j, and

$$\Gamma^w(\tau) = -\sum_{j=0}^{d}\left(\sum_{k=0}^{d}\sum_{m-1}^{n}|\langle x_k|V_m x_j\rangle|^2\int_0^\tau ds\int_0^s dt\; e^{-it(E_\kappa - E_j+\delta_m)}\right)\times|x_j\rangle\langle x_j|, \tag{3.65}$$

where the double integral equals

$$\int_0^\tau ds\int_0^s dt\; e^{-it\alpha} = \begin{cases} \tau^2/2 & \alpha = 0 \\ \frac{1}{\alpha^2}(1-e^{-i\tau\alpha}) - \frac{i}{\alpha}\tau & \alpha \neq 0 \end{cases} \tag{3.66}$$

(v) Note that the formula above shows it is not possible to let the microscopic interaction time τ tend to infinity.

Proof of Corollary 3.2 By Lemma 3.1 above,

$$\left[e^{i\tau t H_P(0)/\lambda^2} PU(t/\lambda^2, 0)P\right] = e^{i\tau t H(0)/\lambda^2}\left[Pe^{-i\tau(H(0)+\lambda W)}P\right]^{t/\lambda^2}, \tag{3.67}$$

where conditions **H1** and **H2** are met and Proposition 3.2 applies. The fact that $\Gamma^w(\tau)$ generates a contraction semigroup in that case stems from the a priori bound, uniform in t, τ, λ,

$$\left\| e^{i\tau t H(0)/\lambda^2} PU(t/\lambda^2, 0)P\right\| \leq 1. \tag{3.68}$$

The expression for $\Gamma^w(\tau)$ comes from the explicit evaluation of (3.60) in our model. $\square$

3.4. Different Time Scales

Looking at the dependence in τ of the result in Corollary 3.2, we observe that we can obtain a different non-trivial effective evolution with our conventional weak limit approach, provided one further makes the time scale $\tau \to 0$ and, at the same time, increases the parameter t to t/τ^2. This yields a macroscopic time scale given by $T = t/(\tau\lambda^2) \to \infty$. We'll come back to this point also, when we deal with the Heisenberg evolution of observables.

Using the first expression (3.24), one immediately gets

$$\lim_{\tau\to 0} \lim_{\substack{\lambda\to 0\\ t/(\tau\lambda)^2\in\mathbb{N}}} \left[e^{i\tau t H(0)/(\tau\lambda)^2} PU(t/(\tau\lambda)^2, 0)P\right] = \lim_{\tau\to 0} e^{t\Gamma w(\tau)/\tau^2} \equiv e^{t\Gamma^1}, \tag{3.69}$$

where

$$\Gamma^1 = {\Gamma_0}^{\#} = \sum_{j=0}^{r} P_j \Gamma_0 P_j, \quad \Gamma_0 = -\frac{1}{2}\sum_{i=1}^{n} V_i {V_i}^*. \tag{3.70}$$

Note that under the hypotheses of Corollary 3.2, the spectral projectors of h_0 and $e^{-i\tau h_0}$ coincide when $\tau \to 0$.

This calls for a redefinition of the scaling, right from the beginning of the calculation, in order to arrive at the same result, without resorting to iterated limits, as above. This is at this point that we need to consider the dependence in τ of the previous steps.

We state below our main theorem regarding this issue in the general Banach space framework under hypotheses **H1** and **H2**. Actually, the application above is a consequence of the theorem to come. The study at positive temperature in Heisenberg picture of the forthcoming Sections will rely on this result as well.

Theorem 3.1. *Suppose Hypotheses **H1** and **H2** hold true and further assume the spectral projectors P_j, $j = 0, \ldots, r$, of $e^{-i\tau H_P(0)}$ coincide with those of $H_P(0)$ on $P\mathcal{B}$. Set $K^{\#} = \sum_{j=0}^{r} P_j K P_j$, for $K \in \mathcal{L}(\mathcal{B})$.*

(A) Then, for any $0 < t_0 < \infty$, there exists $0 < c < \infty$ such that for any $0 \le t < t_0$, the following estimate holds in the limit $\lambda^2\tau \to 0$, $\lambda^2\tau^2 \to 0$, and $t/(\lambda\tau)^2 \in \mathbb{N}$:

$$\left\| e^{iH(0)t/(\lambda^2\tau)} \left[P e^{-i\tau(H(0)+\lambda W)} P\right]^{t/(\lambda\tau)^2} - e^{t e^{i\tau H(0)} G_P(\tau)^{\#}/\tau^2} \right\| \le c(\lambda^2\tau^2 + \lambda^2\tau). \tag{3.71}$$

(B) Then, for any $0 < t_0 < \infty$, there exists $0 < c < \infty$ such that for any $0 \le t \le t_0$ the following estimate holds in the limit $\lambda^2\tau \to 0$, $\lambda^2\tau^2 \to 0$,

and $t/(\lambda\tau)^2 \in \mathbb{N}$*:*

$$\left\| e^{iH(0)t/(\lambda^2\tau)} \left[Pe^{-i\tau(H(0)+\lambda W)}P \right]^{t/(\lambda\tau)^2} - e^{t(W^2)^\#/2} \right\| \leq c(\tau + \lambda^2\tau). \tag{3.72}$$

Remarks

(0) If τ is small enough, the spectral projectors of $e^{-i\tau H_P(0)}$ and $H_P(0)$ on $P\mathcal{B}$ coincide.

(i) If τ is fixed, part (A) of the Theorem coincides with Proposition 3.2 with $\tilde{t} = t/\tau^2$ in place of t.

Proof We only need to consider the case $\tau > 0$ small, where Remark (0) applies. We proceed in two steps, using Lemmas 3.4 and 3.5 in sequence. Let $x = \lambda^2\tau^2$. The expansions provided in Lemma 3.3 yield

$$Pe^{-i\tau(H(0)+\lambda W)}P = e^{-i\tau H_P(0)} + xG_P(\tau)/\tau^2 + O(x^2), \tag{3.73}$$

with $G_P(\tau)/\tau^2 = O(1)$ and reminder uniform in $\tau \to 0$. Hence,

$$\left\| Pe^{-i\tau(H(0)+\lambda W)}P \right\| = 1 + O(x), \quad \text{uniformly in} \quad \tau. \tag{3.74}$$

As $e^{-i\tau H_P(0)} = \sum_j^r e^{-i\tau E_j} P_j$, with P_j independent of τ, the operator $J(\tau)$ defined in (3.39) reads

$$J(\tau) = \sum_{j,k=0}^{r} P_j \frac{G_P(\tau)}{\tau^2} P_k \alpha_{jk}(\tau), \quad \text{where} \quad \alpha_{jk}(\tau) = \begin{cases} \frac{\tau(E_j - E - k)}{e^{i\tau E_j} - e^{-i\tau E_k}} & j \neq k \\ ie^{i\tau E_j} & j = k. \end{cases} \tag{3.75}$$

Hence,

$$\alpha_{jk}(\tau) = i + O(\tau), \ \frac{G_P(\tau)}{\tau^2} = -\frac{W_P^2}{2} + O(\tau) \quad \text{and} \quad J(\tau) = O(1) \ \text{as} \ \tau \to 0. \tag{3.76}$$

Now, using (3.21) with coupling constant x/τ (and the first remark following Lemma 3.3), we can write for x/τ small, uniformly in τ,

$$\begin{aligned} &e^{-i\tau(H_P(0)+\frac{x}{\tau}J(\tau))} \\ &= e^{-i\tau H_P(0)} + \frac{x}{\tau}\left(-ie^{-i\tau H_P(0)} \int_0^\tau e^{isH_P(0)} J(\tau) e^{-isH_P(0)} ds \right) + O((x/\tau)^2\tau^2) \\ &= e^{-i\tau H_P(0)} + x\left(-ie^{-i\tau H_P(0)} \int_0^1 e^{is\tau H_P(0)} J(\tau) e^{-is\tau H_P(0)} ds \right) + O(x^2), \end{aligned} \tag{3.77}$$

where the operator in the bracket above is $O(1)$ as $\tau \to 0$. Hence

$$\left\| e^{-i\tau(H_P(0)+\frac{x}{\tau}J(\tau))} \right\| = 1 + O(x), \quad \text{uniformly in} \quad \tau. \tag{3.78}$$

Thus, we apply Lemma 3.4, to get

$$\| \left[Pe^{-i\tau(H(0)+\lambda W)}P\right]^{\frac{t}{x}} - e^{-i\frac{t}{x}(\tau(H_P(0)+xJ(\tau))}\| = O(x^2), \tag{3.79}$$

as $x \to 0$, and $\frac{x}{\tau} \to 0$, with remainder uniform in τ.

We now turn to the second step. We can write

$$e^{-i\frac{t}{x}(\tau H_P(0)+xJ(\tau))} = e^{-i\frac{t}{\lambda^2\tau}(H_P(0)+\lambda^2\tau J(\tau))} \equiv e^{-i\frac{t}{y}(H_P(0)+yJ(\tau))} \quad \text{with} \quad y = \lambda^2\tau. \tag{3.80}$$

Therefore, by Lemma 3.5 and the last statement of (3.76),

$$e^{i\frac{t}{y}H_P(0)}e^{-i\frac{t}{y}(H_P(0)+yJ(\tau))} - e^{-itJ^{\#}(\tau))} = O(y), \tag{3.81}$$

uniformly in τ. Hence, for any given t_0, we get the existence of a constant $0 < c < \infty$, uniform in τ, such that for all $0 < t \le t_0 < \infty$,

$$\left\|e^{i\frac{t}{\lambda^2\tau}H_P(0)}\left[Pe^{-i\tau(H(0)+\lambda W)}P\right]^{\frac{t}{(\lambda\tau)^2}} - e^{-itJ^{\#}(\tau))}\right\| \le c(\lambda^2\tau + \lambda^2\tau^2), \tag{3.82}$$

as $\lambda^2\tau$ and $\lambda^2\tau^2$ and go to zero in such a way that $t/(\lambda\tau)^2 \in \mathbb{N}$, which is part A of the Theorem. Part B follows from the first statements in (3.76) and of the fact that the projectors P_j's are independent of t. □

As a direct Corollary, we get,

Corollary 3.3. *Let $U(n,0)$ be defined on $\mathcal{H}_0 \otimes \mathcal{H}$, $\mathcal{H}_0$ a separable Hilbert space, by (2.17, 2.16, 2.6), let $P = \mathbb{I} \otimes |\Omega\rangle\langle\Omega|$, and let $\{E_j\}_{j=0,\ldots,r}$ be the eigenvalues of h_0 associated with eigenprojectors $\{P_j\}_{j=0,\ldots,r}$. Then, for any $0 \le t \le t_0$,*

$$\lim_{\substack{\tau\to 0,\lambda^2\tau\to 0\\ t/(\tau\lambda)^2\in\mathbb{N}}} \left[e^{i\tau t H(0)/(\tau\lambda)^2}PU(t/(\tau\lambda)^2,0)P\right] = e^{t\Gamma_0^{\#}}, \tag{3.83}$$

where $\Gamma_0^{\#} = \sum_{j=0}^{r} P_j\Gamma_0 P_j$, and $\Gamma_0 = -\frac{1}{2}\sum_{i=1}^{n} V_i V_i^*$.

4. HEISENBERG REPRESENTATION FOR NON-ZERO TEMPERATURE

From now on, we stick to our model Hamiltonian characterized by hypothesis **H0**. We first express the evolution at positive temperature of observables B of the small system (4.4) after k repeated interactions as the action of the k-th power of an operator $\mathcal{U}_\beta(\lambda,\tau)$ on $\mathcal{H}_0$. This reflects the Markovian nature of our model.

This is done in Proposition 4.1. This allows us to apply Theorem 3.1 again to compute the weak limit in Theorem 4.1. Let us mention here already that we perform a complete analysis of the special case where both the small system and the individual spins of the chain live in $\mathbb{C}^2$ in the last Section of the paper.

Let us define the equilibrium state $\omega(\beta)_N$ of a chain of N spins at inverse temperature β by a tensor product of individual diagonal density matrices of the form

$$r(\beta) = \frac{1}{1+\sum_{j=1}^{n} e^{-\beta\delta_j}} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & e^{-\beta\delta_1} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e^{-\beta\delta_n} \end{pmatrix} = \frac{e^{-\beta\delta a^* a}}{Z(\beta)}, \tag{4.1}$$

in the basis $\{\omega, x_1, \dots, x_n\}$ of $\mathbb{C}_j^{n+1}$, i.e.

$$\omega(\beta)_N = r(\beta) \otimes r(\beta) \otimes \cdots \otimes r(\beta). \tag{4.2}$$

The individual density matrices $r(\beta)$ are defined by Gibbs prescription for the Hamiltonians at each site

$$\sum_{i=1}^{n} \delta_i a_i^* a_i \tag{4.3}$$

corresponding to our model (2.7)

Our spin chain is of finite length N, but, as we will see below, only the first k spins matter to study the time evolution up to time k. This will allow us to take the thermodynamical limit by hand. If ρ is any state on $\mathbb{C}^{d+1}$, the initial state of the small system plus spin chain is $\rho \otimes \omega(\beta)_N$. We shall study the Heisenberg evolution of observables of the form $B \otimes \mathbb{I}_{\mathcal{H}}$, where $B \in M_{d+1}(\mathbb{C})$, defined by

$$B_\beta(k, \lambda, \tau) = \mathrm{Tr}_{\mathcal{H}}((\mathbb{I} \otimes \omega_N(\beta))U(k,0)^{-1}(B \otimes \mathbb{I}_{\mathcal{H}})U(k,0)), \tag{4.4}$$

where, for any $A \in \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H})$,

$$\mathrm{Tr}_{\mathcal{H}}(A) = \left(\sum_S \langle x_i \otimes x_S \mid A x_j \otimes x_S \rangle \right)_{i,j \in \{0,\dots,d\}} \quad \text{with} \quad x_0 = \omega, \tag{4.5}$$

denotes the partial trace taken on the spin variables only. Hence, the expectation in the state ρ of the observable B after k interactions over a time interval of length $k\tau$ with the chain at inverse temperature β is given by

$$\langle B(k, \beta) \rangle_\rho = \mathrm{Tr}_{\mathbb{C}^{d+1}}(\rho B_\beta(k, \lambda, \tau)). \tag{4.6}$$

Remark

In case $\mathcal{H}_0$ is infinite dimensional, the definitions (4.4) and (4.5) hold, *mutatis mutandis*. For instance, consider $B \in \mathcal{L}(\mathcal{H}_0)$ in (4.4), where (4.5) should be read as

$$\mathrm{Tr}_{\mathcal{H}}(A) = \sum_S \langle \mathbb{I}_{\mathcal{H}_0} \otimes x_S \mid A \mathbb{I}_{\mathcal{H}_0} \otimes |x_S\rangle, \tag{4.7}$$

with a slight abuse of notations.

4.1. Markov Properties

Recall that

$$U(k,0)^{-1}(B\otimes \mathbb{I}_{\mathcal{H}})U(k,0) = U_1^* U_2^* \cdots U_k^*(B\otimes \mathbb{I}_{\mathcal{H}})U_k U_{k-1}\cdots U_1, \tag{4.8}$$

where U_j is non-trivial on $\mathbb{C}^{d+1}\otimes \mathbb{C}_j^{n+1}$ only.

Let us specify a bit more the partial trace operator $\mathrm{Tr}_{\mathcal{H}}((\mathbb{I}\otimes \omega_N(\beta))A)$, where A is an operator on $\mathbb{C}^{d+1}\otimes \Pi_{j=1}^N \mathbb{C}_j^{n+1}$.

Lemma 4.1. *Let us denote the matrix elements of A as follows*

$$A_{S,S'}^{i,j} = \langle x_i \otimes X_S \mid A\, x_j \otimes X_{S'}\rangle, \tag{4.9}$$

where i,j belong to $\{0,\ldots,d\}$, and S, S' run over subsets of $\{\{1,\ldots,N\} \times \{1,\ldots,n\}\}^N$ as in (2.3) Then

$$Tr_{\mathcal{H}}((\mathbb{I}\otimes \omega_N(\beta))A)_{i,j} = \sum_S \frac{e^{-\beta\sum_{l=1}^n \delta_l |S|_l}}{(1+\sum_{l=1}^n e^{-\beta\delta_l})^N} A_{S,S}^{i,j} \tag{4.10}$$

where, for

$$S = \{(k_1,i_1),(k_2,i_2),\ldots,(k_m,i_m)\} \subset (\mathbb{N}\times\{1,2,\ldots,n\})^m \tag{4.11}$$

with all $1\le k_j\le N$ distinct and $m=0,\ldots,N$,

$$|S|_l = \#\{k_r\ s.t.\ i_r = l\}. \tag{4.12}$$

Proof Follows directly from

$$\omega_N(\beta)X_S = \frac{\Pi_{r=1}^m e^{-\beta\delta_{i_r}}}{(1+\sum_j e^{-\delta_j\beta})^N} X_S = \frac{e^{-\beta\sum_{l=1}^n \delta_l|S|_l}}{(1+\sum_j e^{-\delta_j\beta})^N} X_S. \tag{4.13}$$

□

We now further compute the action of $U(k,0)$ given by the product of $U_j's$. Let us denote the vectors $\omega\otimes X_S$ and $x_j\otimes X_S$ by $n_0\otimes|n_1,n_2,\ldots,n_N\rangle \equiv n_0\otimes|\vec{n}\rangle$, where $n_0\in\{0,1,\ldots d\}$, and $n_j\in\{0,1,\ldots n\}$, for any $j=1,\ldots N$, with $\omega\simeq 0$ and $x_k\simeq k$ and $X_{\{(1,n_1),\ldots,(N,n_N)\}}\simeq|\vec{n}\rangle$.

Recall that

$$U_j = e^{-i\tau\hat{H}_j} e^{-i\tau\tilde{H}_j}, \tag{4.14}$$

where $e^{-i\tau\hat{H}_j}$ is diagonal. More precisely, with the convention $\delta_0 = 0$,

$$e^{-i\tau\hat{H}_j} n_0\otimes \mid n_1,n_2,\ldots,n_N\rangle = e^{-i\tau\sum_{\substack{k=1\\k\neq j}}^N \delta_{n_k}} n_0\otimes\mid n_1,n_2,\ldots,n_N\rangle. \tag{4.15}$$

Lemma 4.2. *Denoting the k-independent matrix elements of* $e^{-i\tau\tilde{H}_k}|_{\mathbb{C}^{d+1}\otimes\mathbb{C}_k^{n+1}} = \tilde{U}_k|_{\mathbb{C}^{d+1}\otimes\mathbb{C}_k^{n+1}}$ *by*

$$U^{n,n'}_{m,m'} = \langle n \otimes m \mid \tilde{U}_k n' \otimes m' \rangle, \tag{4.16}$$

we have for any $N \geq k$

$$\begin{aligned} &U_k U_{k-1} \cdots U_2 U_1 n_0 \otimes \mid n_1, \ldots, n_N\rangle \\ &= \sum_{\substack{\vec{m_0}\in(0,\ldots,d)^k \\ \vec{m}\in(0,\ldots,n)^k}} e^{-i\tau\varphi(\vec{m},\vec{n})} U^{m_0^k, m_0^{k-1}}_{m_k,n_k} \cdots U^{m_0^2,m_0^1}_{m_2,n_2} U^{m_0^1,n_0}_{m_1,n_1} \\ &\times m_0^k \otimes \mid m_1, m_2, \ldots, m_k, n_{k+1}, \ldots, n_N\rangle, \end{aligned} \tag{4.17}$$

where

$$\varphi(\vec{m},\vec{n}) = \sum_{j=1}^{k} \left(\sum_{j<l\leq N} \delta_{n_l} + \sum_{l<j} \delta_{m_l} \right) \tag{4.18}$$

Proof Consequence of iteration of formulae of the type

$$U_1\ n_0 \otimes \mid n_1, \ldots, n_N\rangle = \sum_{\substack{m_0^1=0,1,\ldots,d \\ m_1=0,1,\cdots,n}} e^{-i\tau\sum_{j>1}\delta_{n_j}} U^{m_0^1,n_0}_{m_1,n_1}\ m_0^1 \otimes \mid m_1, n_2, n_3, \ldots, n_N\rangle. \tag{4.19}$$

□

A consequence of these formulae is that we can consider spin chains consisting in k spins only:

Lemma 4.3. *For any* $N \geq k$,

$$\begin{aligned} &Tr_{\mathcal{H}}(\mathbb{I} \otimes \omega_N(\beta) U_1^* U_2^* \cdots U_k^*(B \otimes \mathbb{I}_{\mathcal{H}}) U_k U_{k-1} \cdots U_1) \\ &\quad = Tr_{\mathcal{H}}(\mathbb{I} \otimes \omega_k(\beta) U_1^* U_2^* \cdots U_k^*(B \otimes \mathbb{I}_{\mathcal{H}}) \\ &\quad \times U_k U_{k-1} \cdots U_1) \end{aligned} \tag{4.20}$$

Proof Obvious from the tensor product structure of $\omega_N(\beta)$. □

To proceed, let us adopt the following block notation

$$U = e^{-i\tau(H(0)+\lambda W)} = \begin{pmatrix} U_{0,0} & U_{0,1} & \cdots & U_{0,n} \\ U_{1,0} & U_{1,1} & \cdots & U_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ U_{n,0} & U_{n,1} & \cdots & U_{n,n} \end{pmatrix} \tag{4.21}$$

where

$$U_{m,m'} = \begin{pmatrix} U^{0,0}_{m,m'} & U^{0,1}_{m,m'} & \cdots & U^{0,d}_{m,m'} \\ U^{1,0}_{m,m'} & U^{1,1}_{m,m'} & \cdots & U^{1,d}_{m,m'} \\ \vdots & \vdots & \ddots & \vdots \\ U^{d,0}_{m,m'} & U^{d,1}_{m,m'} & \cdots & U^{d,d}_{m,m'} \end{pmatrix}. \tag{4.22}$$

In terms of the notations of the previous Section,

$$U = \begin{pmatrix} PUP & PUQ \\ QUP & QUQ \end{pmatrix}, \tag{4.23}$$

we have the identifications

$$PUP \simeq U_{0,0}, \quad QUQ \simeq \begin{pmatrix} U_{1,1} & \cdots & U_{1,n} \\ \vdots & \ddots & \vdots \\ U_{n,1} & \cdots & U_{n,n} \end{pmatrix},$$

$$PUQ \simeq (U_{0,1} \cdots U_{0,n}), \quad QUP \simeq (U_{1,0} \cdots U_{n,0})^T. \tag{4.24}$$

Let us finally denote the inverse of $U = (U^{n,n'}_{m,m'})$ by

$$V = \left(V^{n,n'}_{m,m'}\right) = U^{-1} = \left({U^{-1}}^{\,n,n'}_{\,m,m'}\right) \in M_{(1+d)(1+n)}(\mathbb{C}), \tag{4.25}$$

so that we have for any m and n

$$U^*_{n,m} = V_{m,n} \in M_{1+d}(\mathbb{C}). \tag{4.26}$$

With these notations, we have

Lemma 4.4. *The matrix elements of $U(k,0)^{-1}(B \otimes \mathbb{I}_{\mathcal{H}})U(k,0)$ in the orthonormal basis $\{n_0 \otimes \mid n_1, \ldots, n_k\rangle\} = \{n_0 \otimes |\vec{n}\rangle\}$ read*

$$\begin{aligned} &\langle \tilde{n}_0 \otimes \vec{\tilde{n}} | (U_k \cdots U_1)^* B \otimes \mathbb{I}_{\mathcal{H}} (U_k \cdots U_1) n_0 \otimes \vec{n}\rangle \\ &= e^{-i\tau(\varphi(0,\vec{n})-\varphi(0,\vec{\tilde{n}}))} \sum_{\vec{m}\in\{0,\ldots,n\}^k} \left(V_{\tilde{n}_1,m_1} \cdots V_{\tilde{n}_k,m_k} B U_{m_k,n_k} \cdots U_{m_1,n_1}\right)^{\tilde{n}_0,n_0} \end{aligned} \tag{4.27}$$

Proof Expand the products and make use of Lemma 4.2 and (4.18). □

The above Lemmas and (4.4) lead us to the study of the matrix in $M_{d+1}(\mathbb{C})$

$$B_\beta(k,\lambda,\tau) = \sum_{\substack{\vec{n}=(n_1,\dots,n_k)\\ \vec{m}=(m_1,\cdots m_k)}} \frac{e^{-\beta\sum_{l=0}^n \delta_l|\vec{n}|_l}}{(1+\sum_{j=1}^n e^{-\delta_j\beta})^k} V_{n_1,m_1}\cdots V_{n_k,m_k} B U_{m_k,n_k}\cdots U_{m_1,n_1} \tag{4.28}$$

in various limiting cases as λ and/or τ go to zero, with the notation

$$|\vec{n}|_l = \sharp\{n_r \quad \text{s.t.} \quad n_r = l\} = |S|_l. \tag{4.29}$$

We introduce operators on the Hilbert space $M_{d+1}(\mathbb{C})$ equipped with the scalar product $(A|B\rangle = \mathrm{Tr}(A^*B)$, for any $A, B \in M_{d+1}(\mathbb{C})$ by

$$\mathcal{U}_{m,m'}(A) := V_{m'm} A U_{m,m'}, \quad (m,m') \in \{0,1,\dots,n\}^2. \tag{4.30}$$

These operators are linear and one has with respect to the above scalar product,

$$\mathcal{U}^*_{m,m'}(\cdot) = (V_{m',m}\cdot U_{m,m'})^* = U_{m,m'}\cdot V_{m',m}. \tag{4.31}$$

The composition of such operators will be denoted as follows

$$\mathcal{U}_{m',n'}\mathcal{U}_{m,n}(A) = V_{n'm'}V_{n,m}AU_{m,n}U_{m',n'}. \tag{4.32}$$

We are now in a position to express the Markovian nature of the evolution of our observables:

Proposition 4.1. *In terms of the operators defined above, we can write*

$$\begin{aligned} B_\beta(k,\lambda,\tau) = {} & \frac{1}{(1+\sum_{j=1}^n e^{-\delta_j\beta})^k}\big(\mathcal{U}_{0,0} + e^{-\beta\delta_1}\mathcal{U}_{0,1} + \cdots + e^{-\beta\delta_n}\mathcal{U}_{0,n} \\ & + \mathcal{U}_{1,0} + e^{-\beta\delta_1}\mathcal{U}_{1,1} + \cdots + e^{-\beta\delta_n}\mathcal{U}_{1,n} \\ & + \mathcal{U}_{n,0} + e^{-\beta\delta_1}\mathcal{U}_{n,1} + \cdots + e^{-\beta\delta_n}\mathcal{U}_{n,n}\big)^k (B) \\ \equiv {} & \mathcal{U}_\beta(\lambda,\tau)^k(B). \end{aligned} \tag{4.33}$$

Proof By definition of $\mathcal{U}_{m,n}$ we have

$$B_\beta(k,\lambda,\tau) = \sum_{\substack{\vec{n}=(n_1,\dots,n_k)\\ \vec{m}=(m_1,\dots m_k)}} \frac{e^{-\beta\sum_{l=1}^k \delta_{n_l}}}{(1+\sum_{j=1}^n e^{-\delta_j\beta})^k}\mathcal{U}_{m_1,n_1}\cdots\mathcal{U}_{m_k,n_k}(B). \tag{4.34}$$

Furthermore introducing $\mathcal{Y}_{m,n} = e^{-\delta_n \beta}\mathcal{U}_{m,n}$, we get

$$B_\beta(k, \lambda, \tau) = \frac{1}{(1 + \sum_{j=1}^n e^{-\delta_j \beta})^k} \sum_{\substack{\vec{n}=(n_1,\dots,n_k)\\ \vec{m}=(m_1,\dots,m_k)}} \mathcal{Y}_{m_1,n_1} \cdots \mathcal{Y}_{m_k,n_k}(B). \tag{4.35}$$

There are $(n+1)^2$ distinct operators $\mathcal{Y}_{m,n'}$ in that expression, and the set of vectors $\vec{n}, \vec{m}$ in the sum yields all different ways of composing k of them. Therefore

$$B_\beta(k, \lambda, \tau) = \frac{1}{(1 + \sum_{j=1}^n e^{-\delta_j \beta})^k} \times (\mathcal{Y}_{0,0} + \cdots + \mathcal{Y}_{0,n} + \cdots + \mathcal{Y}_{n,0} + \cdots, \mathcal{Y}_{n,n})^k(B). \tag{4.36}$$

□

Remark

The formula of Proposition 4.1 holds if $\mathcal{H}_0$ is a separable Hilbert space, provided the decomposition of operators A in (4.21) is interpreted as $A_{pq} \in \mathcal{L}(\mathcal{H}_0)$, $q, p \in \{1, \dots, n\}$, with

$$A_{pq} = \mathbb{I}_{\mathcal{H}_0} \otimes |p\rangle\langle p| A \mathbb{I}_{\mathcal{H}_0} \otimes |q\rangle\langle q|, \tag{4.37}$$

and the identification $\mathcal{H}_0 \otimes \mathbb{C}|q\rangle \simeq \mathcal{H}_0$, for all q.

4.2. Weak Limit in the Heisenberg Picture

The λ-dependence in $B_\beta(k, \lambda, \tau)$ comes from the definition

$$U = U_\tau(\lambda) = e^{-i\tau(H(0)+\lambda W)}, \tag{4.38}$$

which implies that the $\mathcal{U}_{n,m}$'s depend on λ as well, in an analytic fashion, and will be denoted $\mathcal{U}_{n,m}(\lambda)$. Expliciting the λ dependence in $B_\beta(k, \lambda, \tau)$, the weak limit corresponds to taking $k = t/\lambda^2$ and computing the behavior of $B_\beta(t/\lambda^2, \lambda, \tau)$, as $\lambda \to 0$ (keeping τ fixed). We shall use the same strategy as in the previous Section and Lemma 3.1 to identify the weak limit by means of perturbation theory. We shall also eventually consider the possibility of letting $\tau \to 0$, therefore we explicit the behavior in τ of the expansions below.

Consequently, with (4.24) and Corollary 3.1, we get

Lemma 4.5. *Let U be given by (4.38), with $H(0)$, W self adjoint and satisfying **H1**, and further assume $H(0)$ is diagonal with respect to the basis (3.5). If $\mathcal{U}_{m,m'}(\lambda)$ is defined by (4.30), as $\lambda \to 0$, we get the expansions*

$$\mathcal{U}_{0,0}(\lambda) = \mathcal{U}_{0,0}(0) + \lambda^2 \mathcal{U}^{(2)}_{0,0} + O(\lambda^4\tau^4) \tag{4.39}$$

$$\mathcal{U}_{m,m'}(\lambda) = \mathcal{U}_{m,m'}(0) + \lambda^2 \mathcal{U}^{(2)}_{m,m'} + O(\lambda^4\tau^4), \quad m, m' \geq 1 \tag{4.40}$$

$$\mathcal{U}_{0,m}(\lambda) = \lambda^2 \mathcal{U}^{(1)}_{0,m} + O(\lambda^4\tau^4), \quad m \geq 1 \tag{4.41}$$

$$\mathcal{U}_{m,0}(\lambda) = \lambda^2 \mathcal{U}^{(1)}_{m,0} + O(\lambda^4\tau^4), \quad m \geq 1 \tag{4.42}$$

where, for all $0 \leq m, m' \leq n$

$$\mathcal{U}_{m,m'}(0)(B) = \delta_{m,m'} e^{i\tau H_{m,m}(0)} B e^{-i\tau H_{m,m}(0)}, \tag{4.43}$$

$$\mathcal{U}^{(2)}_{m,m'}(B) = \delta_{m,m'} \big(G_{m,m}(-\tau) B e^{-i\tau H_{m,m}(0)} + e^{i\tau H_{m,m}(0)} B G_{m,m}(\tau)\big), \tag{4.44}$$

and, for all $1 \leq m$,

$$\mathcal{U}^{(1)}_{0,m}(B) = F_{m,0}(-\tau) B F_{0,m}(\tau), \tag{4.45}$$

$$\mathcal{U}^{(1)}_{m,0}(B) = F_{0,m}(-\tau) B F_{m,0}(\tau). \tag{4.46}$$

This Lemma allows us to perform the analysis of the operator defined in Proposition 4.1

$$\mathcal{U}_\beta(\lambda, \tau) = \mathcal{Z}(\beta)^{-1} \sum_{\substack{0\leq m\leq n\\ 0\leq l\leq n}} \mathcal{U}_{l,m}(\lambda) e^{-\delta_m \beta}, \quad \text{as} \quad \lambda \to 0, \tag{4.47}$$

with the convention $\delta_0 = 0$ and $\mathcal{Z}(\beta) = \sum_{j=0}^n e^{-\delta_j \beta}$. Recall that

$$B_\beta(k, \lambda, \tau) = \mathcal{U}_\beta(\lambda, \tau)^k(B). \tag{4.48}$$

Moreover, using the fact, see (3.4),

$$H_{m,m}(0) = H_{0,0}(0) + \delta_m \simeq h_0 + \delta_m, \tag{4.49}$$

we get for all $0 \leq m \leq n$

$$\mathcal{U}_{m,m}(0)(B) = \mathcal{U}_{0,0}(0)(B) \simeq e^{i\tau h_0} B e^{-i\tau h_0} = e^{i\tau[h_0,\cdot]}(B). \tag{4.50}$$

We have thus shown the

Lemma 4.6. *Assume the hypotheses of Lemma 4.5. Then*

$$\mathcal{U}_\beta(\lambda, \tau) = \mathcal{U}_{0,0}(0) + \frac{\lambda^2}{\mathcal{Z}(\beta)} \left[\sum_{m=1}^n \left\{ e^{-\beta\delta_m} \left(\mathcal{U}^{(1)}_{0,m} + \mathcal{U}^{(2)}_{m,m} \right) + \mathcal{U}^{(1)}_{m,0} \right\} + \mathcal{U}^{(2)}_{0,0} \right]$$

$$+ O(\lambda^4\tau^4) \equiv \mathcal{U}_{0,0}(0) + \lambda^2 \mathcal{Z}(\beta)^{-1} T_\beta + O(\lambda^4\tau^4), \tag{4.51}$$

with $T_\beta = T_\beta(\tau) = O(\tau^2)$.

The above operator enjoys the following symmetry property

Lemma 4.7. *For any $B \in M_{d+1}(\mathbb{C})$,*

$$Tr(BT_\beta(B^*)) = \overline{Tr(B^* T_\beta(B))}. \tag{4.52}$$

Proof Due to

$$F_{n,m}(-\tau) = F_{m,n}(\tau)^*, \quad m \neq n \tag{4.53}$$

$$G_{n,n}(-\tau) = G_{n,n}(\tau)^* \tag{4.54}$$

and to the structure of T_β, the result will be proven once we show that for all $A, B, C \in M_{d+1}(\mathbb{C})$

$$\overline{\mathrm{Tr}(B^*ABC + B^*C^*BA^*)} = \mathrm{Tr}(BAB^*C + BC^*B^*A^*). \tag{4.55}$$

But this follows from $\mathrm{Tr}B = \mathrm{Tr}B^T$, where $\cdot^T$ denotes the transpose, and from the cyclicity of the trace again. □

Recall also the property

$$\mathcal{U}_\beta(\lambda, \tau)(\mathbb{I}) = \mathbb{I} \;\Rightarrow\; T_\beta(\mathbb{I}) = 0, \tag{4.56}$$

and the fact that in case the spectrum $\{E_j\}_{j=0,\dots,d}$ of h_0 is non-degenerate and $\{|x_j\rangle\}_{j=0,\dots,d}$ denotes the corresponding eigenvectors, the unitary $\mathcal{U}_{0,0}(0)$ has degenerate spectrum:

$$\mathcal{U}_{0,0}(0)(|x_j\rangle\langle x_k|) = e^{i\tau(E_j - E_k)}|x_j\rangle\langle x_k|, \quad \forall\, 0 \leq j, \quad k \leq d. \tag{4.57}$$

That is, $\sigma(\mathcal{U}_{0,0}(0)) = \{e^{i\tau(E_j - E_k)}\}_{0\leq j,k\leq d}$, so that 1 is $d+1$ times degenerate at least.

We are in the same position as in the proof of Proposition (3.2). Therefore, we can compute the weak limit from Proposition 3.1 immediately to get the following

Theorem 4.1. *Let $\mathcal{U}_\beta(\lambda, \tau)$ be given by (4.47), and $\mathcal{U}_{0,0}(0)$, T_β by (4.51). Let $\{e^{i\tau\Delta_l}\}_{i=1,\dots,r}$ be the set of distinct eigenvalues of $\mathcal{U}_{0,0}(0)$ and denote by P_l the corresponding orthogonal projectors. Then*

$$\begin{aligned}&\lim_{\substack{\lambda\to 0\\ t/\lambda^2\in\mathbb{N}}} \mathcal{U}_{0,0}(0)^{-t/\lambda^2} B_\beta(t/\lambda^2, \lambda, \tau)\\ &\quad = \lim_{\substack{\lambda\to 0\\ t/\lambda^2\in\mathbb{N}}} \mathcal{U}_{0,0}(0)^{-t/\lambda^2}\mathcal{U}_\beta(\lambda, \tau)^{t/\lambda^2}(B) = e^{t\Gamma^w_\beta}(B),\end{aligned} \tag{4.58}$$

where

$$\Gamma^w_\beta(B) = \frac{1}{\mathcal{Z}(\beta)}(\mathcal{U}_{0,0}(0)^{-1}T_\beta)^{\#}(B), \tag{4.59}$$

with # corresponding to the set of projectors $\{P_l\}_{l=1,\dots,r}$.

Remarks

(0) In order to make the generator Γ_β^w completely explicit, one needs to analyse the properties of T_β, i.e. of the operators V_j defining the coupling, within the eigenspaces of $\mathcal{U}_{0,0}(0)$. A non trivial example is worked out in Sec. 6, see Proposition 6.1

(i) The degeneracy of the eigenvalue 1 of $\mathcal{U}_{0,0}(0)$ is responsible for the existence of a nontrivial invariant sub-algebra of observables which is the commutant of h_0.

As in Sec. 2, we generalize our result to the regime $\lambda^2\tau \to 0$, $\tau \to 0$, by switching to the macroscopic time scale $T = t/(\lambda^2\tau) \to \infty$. We first compute

$$\begin{aligned}\Gamma_\beta = \lim_{\tau\to 0} \frac{\mathcal{U}_{0,0}(0)^{-1}T_\beta}{\mathcal{Z}(\beta)\tau^2}(B) = -\frac{1}{2\mathcal{Z}(\beta)}\big(W^2{}_{0,0}B + B\,W^2{}_{0,0}\big)\\ + \frac{1}{\mathcal{Z}(\beta)}\sum_{m=1}^{n}\left\{e^{-\delta_m\beta}\left(W_{m,0}BW_{0,m} - \frac{1}{2}\big(W^2{}_{m,m}B + B\,W^2{}_{m,m}\big)\right)\right.\\ \left. + W_{0,m}BW_{m,0}\right\},\end{aligned} \tag{4.60}$$

which, using the following formulas for $m \geq 1$

$$W_{0,m} = V_m^*, \quad W_{m,0} = V_m, \quad W^2{}_{m,m} = V_mV_m^*, \quad W^2{}_{0,0} = \sum_{j=1}^{n} V_j^*V_j, \tag{4.61}$$

to express the operators $W_{mm'}$ in terms of V_m, eventually becomes

$$\begin{aligned}\Gamma_\beta(B) = \frac{1}{\mathcal{Z}(\beta)}\sum_{m=1}^{n} e^{-\beta\delta_m}\left(V_mBV_m^* - \frac{1}{2}(V_mV_m^*B + BV_mV_m^*)\right)\\ + V_m^*BV_m - \frac{1}{2}(V_m^*V_mB + BV_m^*V_m).\end{aligned} \tag{4.62}$$

We note here that this operator has the form of the dissipative part of a Lindblad generator. We'll come back to this operator Γ_β in connection to the modelization in terms of Quantum Noises proposed in Ref. 13 and Ref. 3, in the next Section.

Corollary 4.1. *Assume the hypotheses of Theorem 4.1. Then with $t/(\tau\lambda)^2 = k \in \mathbb{N}$,*

$$\begin{aligned}&\lim_{\substack{\tau\to 0,\lambda^2\tau\to 0\\ t/(\tau\lambda)^2\in\mathbb{N}}} \mathcal{U}_{0,0}(0)^{-t/(\tau\lambda)^2} B_\beta(t/(\tau\lambda)^2, \lambda, \tau))\\ &= \lim_{\substack{\tau\to 0,\lambda^2\tau\to 0\\ t/(\tau\lambda)^2\in\mathbb{N}}} \mathcal{U}_{0,0}(0)^{-t/(\tau/\lambda)^2}\mathcal{U}_\beta(\lambda,\tau)^{t/\lambda^2}(B) = e^{t\Gamma_\beta^\#}(B),\end{aligned} \tag{4.63}$$

were $\Gamma_\beta(B)$ is defined in (4.62).

Proof We can simply repeat the arguments of the proof Theorem 3.1 once we note the following facts: i) The operator $\mathcal{U}_{0,0}(0) = e^{i\tau[h_0,\cdot]}$ is unitary on $M_{d+1}(\mathbb{C})$, with spectral projectors that are independent of τ as $\tau \to 0$ and eigenvalues of the form $e^{i\tau\Delta_j}$. ii) Introducing $x = (\lambda\tau)^2$, (4.51) states that uniformly in τ,

$$\mathcal{U}_\beta(\lambda,\tau) = \mathcal{U}_{0,0}(0) + xT_\beta(\tau)/(\tau^2\mathcal{Z}(\beta)) + O(x^2), \tag{4.64}$$

where $T_\beta(\tau)/\tau^2 \to \Gamma_\beta$ as $\tau \to 0$. □

4.3. Evolution of States

Let us close this Section by briefly recalling some consequences of these results about the evolution of states, i.e. trace one positive matrices. This is conveniently done in our setup by using duality with respect to the scalar product $\langle A|B\rangle = \mathrm{Tr}(A^*B)$.

If Γ is the generator of the dynamics of observables, B is an observable and ρ is a state, then for any $t \in \mathbb{R}$,

$$\mathrm{Tr}(\rho e^{t\Gamma}(B)) = \mathrm{Tr}(e^{t\Gamma_*}(\rho)B) \tag{4.65}$$

where the generator of the dynamics of the states is Γ_* such that for all states ρ and observables B,

$$\mathrm{Tr}((\Gamma_*(\rho))^*B) = \langle\rho \mid \Gamma(B)\rangle = \langle\Gamma^*\rho \mid B\rangle. \tag{4.66}$$

In the particular case where the observables $P_{jk} = |x_j\rangle\langle x_k|$, with the notations of (4.57), form an orthonormal basis of eigenvectors of the restricted uncoupled evolution $\mathcal{U}_{0,0}$, the corresponding eigenprojectors are denoted by Π_{jk} and act as

$$\Pi_{jk}(B) = P_{jk}\mathrm{Tr}(\mid x_k\rangle\langle x_j \mid B) = P_{jk}\langle x_j \mid Bx_k\rangle_{\mathcal{H}_0}, \tag{4.67}$$

where the subscript $\mathcal{H}_0$ denotes the scalar product within $\mathcal{H}_0$. Hence, the # operation on the operator Γ with respect to the projectors Π_{jk} is given by

$$\Gamma^\#(B) = \sum_{j,k}\Pi_{jk}\Gamma\Pi_{jk}(B) = \sum_{j,k}|x_j\rangle\langle x_k \mid \langle x_j \mid \Gamma(|x_j\rangle\langle x_k|)x_k\rangle_{\mathcal{H}_0}\langle x_j \mid Bx_k\rangle_{\mathcal{H}_0}. \tag{4.68}$$

Therefore, one computes that the corresponding generator of states, $(\Gamma^\#)_*$ is given by

$$\Gamma^\#_*(\rho) = \sum_{j,k} \mid x_j\rangle\langle x_k|\langle x_k \mid \Gamma(|x_j\rangle\langle x_k|)^* x_j\rangle_{\mathcal{H}_0}\langle x_j \mid \rho x_k\rangle_{\mathcal{H}_0}. \tag{4.69}$$

Consequently,

$$(\Gamma^{\#})_{*} = \sum_{j,k} \Pi_{jk} \Gamma_{*} \Pi_{jk} = (\Gamma_{*})^{\#}. \tag{4.70}$$

We note that states defined as functions of the Hamiltonian h_0 of the small system form an invariant subspace of sets whose Markovian dynamics is characterized by the scalars $\{\langle x_j \mid \Gamma(|x_j\rangle\langle x_j|) x_j\rangle_{\mathcal{H}_0}\}_{j=0,\dots,d}$.

5. BEYOND THE PERTURBATIVE REGIME: $\lambda^2\tau = 1$

We consider here the regime $\lambda^2\tau = 1$, and $\tau \to 0$ used in Ref. 3 in their construction of the field of quantum noises. It can be viewed as a regime where the weak limit scaling holds at the microscopic level, while, at the macroscopic level, $T = t/(\tau\lambda^2)$ is kept finite.

As we saw in Corollaries 3.3 and 4.1 in the Schrödinger and Heisenberg pictures respectively, the small parameter that allows to make use of perturbation theory to compute the effective evolution is the combination $\lambda^2\tau$. Therefore, we have to resort to a different technique since our scaling imposes a non-perturbative regime. Our main tool will be Chernoff's Theorem as we now explain.

5.1. Schrödinger Evolution

Let us start with the Schrödinger effective evolution under the following assumptions:

H3: Hypothesis **H1** holds with $\mathcal{B}$ a Hilbert space and P, $H(\lambda) = H(0) + \lambda W$ self-adjoint.

In the scaling adopted here, the number of interactions n has to grow like $n = t/\tau$. This is in keeping with the fact that in all cases considered so far, $n = t/(\lambda\tau)^2 = t/\tau$. Note that the macroscopic time $T = \tau n = t$ is finite here. Therefore, according to the analysis of Sec. 3, we are led to study.

$$PU(t/\tau, 0)P = \left[Pe^{-i(\tau H(0)+\sqrt{\tau}W)}P\right]^{t/\tau}, \quad \text{as} \quad \tau \to 0, \quad t/\tau \in \mathbb{N}^*. \tag{5.1}$$

This limit is easily computed by applying the following version of Chernoff's Theorem, see e.g.,[5,7,14] which suffices for our purpose:

Theorem 5.1. *Let $S(\tau)$ defined on a Banach space $\mathcal{B}$ be such that $S(0) = \mathbb{I}$, and $\|S(\tau)\| \leq 1$ for all $\tau \geq 0$. If, $\lim_{\tau\to 0} \tau^{-1}(S(\tau) - \mathbb{I}) = \Gamma$ in the strong sense exists in $\mathcal{L}(\mathcal{B})$ and generates a contraction semi-group, then*

$$s - \lim S(t/n)^n = e^{t\Gamma}. \tag{5.2}$$

Now, it is easily checked that

$$S(\tau) := Pe^{-i(\tau H(0)+\sqrt{\tau}W)}P \quad \text{on the subspace} \quad P\mathcal{B} \tag{5.3}$$

satisfies the first requirements. Then, by expanding the exponent and making use of the properties of $H(0)$ and W, we can write

$$S(\tau) = \left(\mathbb{I} - i\tau H(0)_P - \frac{\tau}{2}(W^2)_P + O(\tau^2)\right). \tag{5.4}$$

It thus implies

$$S'(\tau)|_{\tau=0} = -iH(0)_P - \frac{(W^2)_P}{2} = \Gamma \in \mathcal{L}(P\mathcal{B}). \tag{5.5}$$

Now Γ is dissipative, since $\forall\varphi \in P\mathcal{B}$

$$\Re\langle\varphi \mid \Gamma\varphi\rangle = -\Re\langle\varphi \mid PWQWP\varphi\rangle/2 = -\|QWP\varphi\|_{\mathcal{B}}/2 \leq 0. \tag{5.6}$$

Hence, by Lumer-Phillips, see, [14] Γ generates a contraction semigroup. Therefore

Theorem 5.2. *Under the hypothesis **H3**, for any $t > 0$ fixed,*

$$s - \lim_{\substack{\tau\to 0\\ t/\tau\in\mathbb{N}}} PU(t/\tau, 0)P = s - \lim_{\substack{\tau\to 0\\ t/\tau\in\mathbb{N}}} \left[Pe^{-i(\tau H(0)+\sqrt{\tau}W)P}\right]^{t/\tau} = e^{-t(iH(0)_P+\frac{(w^2)_P}{2})}. \tag{5.7}$$

Remark. Specializing to our model Hamiltonian, we get that the effective dynamics on $P\mathcal{B}$ is

$$e^{-t(ih_0+\frac{1}{2}\sum_j V_j^* V_j)}. \tag{5.8}$$

Apart from the self-adjoint part h_0 stemming from the uncoupled evolution, the main difference with respect to the corresponding weak coupling result in Corollary 3.3, lies in the absence of the # operation on the dissipative part $\frac{1}{2}\sum_j V_j^* V_j$ of the generator. This prevents the spectral subspaces of h_0 from being invariant under the effective dynamics.

5.2. Heisenberg Evolution

Let us now turn to the more interesting case of the Heisenberg dynamics of observables when the spins are at equilibrium at inverse temperature β. We assume the general hypothesis **H0**, i.e. we stick to our matrix model, even though certain results below hold for more general situations.

The analysis of Sec. 4 shows that the evolution of an observable $B \in M_{d+1}(\mathbb{C})$ after k repeated interactions reads

$$B \mapsto B_\beta(k, \lambda, \tau) = \mathcal{U}_\beta(\lambda, \tau)^k(B) \tag{5.9}$$

with $\mathcal{U}_\beta(\lambda,\tau)$ defined by (4.47), where we explicited the dependence in τ in the notation. We want to apply Chernoff's Theorem again to the operator valued function $\tau \mapsto \mathcal{U}_\beta(1/\sqrt{\tau},\tau)$ on $\mathcal{L}(M_{d+1}(\mathbb{C}))$. In order to check the first hypotheses we recall the formula (see (4.10))

$$\begin{aligned}\mathcal{U}_\beta(\lambda,\tau)(B) &= \mathrm{Tr}_{\mathcal{H}}((\mathbb{I}\otimes\omega_1(\beta))U^{-1}(1,0)(B\otimes\mathbb{I})U(1,0)) \\ &= \sum_{q=0}^{n}\frac{e^{-\beta\delta_q}}{\mathcal{Z}(\beta)}\mathbb{B}(\tau)_{qq},\end{aligned} \tag{5.10}$$

where $\mathbb{B}(\tau)_{qq} = (U^{-1}(1,0)(B\otimes\mathbb{I})U(1,0))_{qq} = P_qU^{-1}(1,0)(B\otimes\mathbb{I})U(1,0)P_q$ according to the block notation (4.21), with the corresponding orthogonal projectors P_q. Identifying $P_q\mathbb{C}^{(n+1)(d+1)}$ with $\mathcal{H}_0=\mathbb{C}^{d+1}$, we deduce from the above formula that $\mathcal{U}_\beta(\lambda,\tau)$ is a contraction for any value of the parameters:

$$\begin{aligned}\|\mathcal{U}_\beta(\lambda,\tau)(B)\|\mathcal{H}_0 &\leq \sum_{q=0}^{n}\frac{e^{-\beta\delta_q}}{\mathcal{Z}(\beta)}\|\mathbb{B}(\tau)_{qq}\|_{\mathcal{H}_0} \\ &\leq \sum_{q=0}^{n}\frac{e^{-\beta\delta_q}}{\mathcal{Z}(\beta)}\|P_qU^{-1}(1,0)(B\otimes\mathbb{I})U(1,0)P_q\|_{\mathbb{C}^{(n+1)(d+1)}} \\ &\leq \sum_{q=0}^{n}\frac{e^{-\beta\delta_q}}{\mathcal{Z}(\beta)}\|(B\otimes\mathbb{I})\|_{\mathbb{C}^{(n+1)(d+1)}} = \|B\|_{\mathcal{H}_0}.\end{aligned} \tag{5.11}$$

Moreover, $\mathcal{U}_\beta(1/\sqrt{\tau},\tau)|_{\tau=0}=\mathbb{I}$, so we are left with the computation of the derivative w.r.t. τ at the origin. This involves the control of the operator $U_\tau(\lambda)$ (3.20) as $\tau\to 0$ and $\lambda = 1/\sqrt{\tau}\to\infty$, as in the previous paragraph. Let us get estimates in a more systematic way than above. So far, all our estimates are derived for both λ and τ going to zero or at most finite. However, the expansion of $U_\tau(\lambda)$ in powers of λ is convergent, with τ dependent coefficients we control sufficiently well. Indeed, (3.33) yields

$$U_\tau(\lambda) = e^{-i\tau H(0)}\Theta(\lambda,\tau) = \sum_{n\geq 0} e^{-i\tau H(0)}\Theta_n(\lambda,\tau), \tag{5.12}$$

where Θ_n contains n operators W and satisfies

$$\|\Theta_n(\lambda,\tau)\| = O((\tau\lambda)^n/n!). \tag{5.13}$$

Using the fact that $(\lambda\tau)^n = \tau^{n/2}\to 0$ and that W is off-diagonal with respect to P and Q, we get that the replacement of λ by $1/\sqrt{\tau}$ doesn't spoil the estimates as $\tau\to 0$ given in Proposition 4.1 and Lemma 4.5. Those together with the computation (4.62) yield

$$\mathcal{U}_\beta(1/\sqrt{\tau},\tau)(B) = e^{i\tau h_0}Be^{-i\tau h_0} + (\mathcal{Z}(\beta)\tau)^{-1}T_\beta(\tau)(B) + O(\tau^2)$$

$$\equiv e^{i\tau h_0} B e^{-i\tau h_0} + \tau \Gamma_\beta(B) + O(\tau^2), \tag{5.14}$$

where, see (4.62),

$$\Gamma_\beta(B) = \frac{1}{\mathcal{Z}(\beta)} \sum_{m=1}^{n} e^{-\beta\delta_m} \left(V_m B V_m^* - \frac{1}{2}(V_m V_m^* B + B V_m V_m^*) \right.$$

$$\left. + V_m^* B V_m - \frac{1}{2}(V_m^* V_m B + B V_m^* V_m) \right). \tag{5.15}$$

Hence, the derivative at the origin exists and is given by

$$\mathcal{U}_\beta(1/\sqrt{\tau}, \tau)'(B)|_{\tau=0} = i[h_0, B] + \Gamma_\beta(B). \tag{5.16}$$

We recognize at once that $\Gamma_\beta(B)$ is the dissipative part of a Lindblad operator of the form

$$\sum_{j=1}^{2m} L_j B L_j^* - \frac{1}{2}(L_j L_j^* B + B L_j L_j^*) \tag{5.17}$$

with

$$L_j = \frac{e^{-\beta\delta_j/2}}{\sqrt{\mathcal{Z}(\beta)}} V_j, \quad 1 \le j \le m \quad \text{and} \quad L_j = \frac{1}{\sqrt{\mathcal{Z}(\beta)}} V_j^*, \quad m+1 \le j \le 2m. \tag{5.18}$$

By the Theorem of Lindblad, see e.g.,[1] we know that

$$i[h_0, B] + \Gamma_\beta(B) \tag{5.19}$$

generates a completely positive semigroup of contractions. Therefore, we are in a position to apply Chernoff's theorem to eventually get

Theorem 5.3. *Assume hypothesis **H0** where $\mathcal{H}_0$ is a separable Hilbert space and h_0, the V_j's and B are bounded on $\mathcal{H}_0$. Let $B_\beta(t/\tau, 1\sqrt{\tau}, \tau)$ be defined by (4.4), $\mathcal{U}_\beta(\lambda, \tau)$ is defined by proposition 4.1 and the Remark following it. Then*

$$s - \lim_{\substack{\tau\to 0\\ t/\tau\in\mathbb{N}}} B_\beta(t/\tau, 1/\sqrt{\tau}, \tau) = s - \lim_{\substack{\tau\to 0\\ t/\tau\in\mathbb{N}}} \mathcal{U}_\beta(1/\sqrt{\tau}, \tau)^{t/\tau}(B) = e^{t(i[h_0,\cdot]+\Gamma_\beta(\cdot))}(B) \tag{5.20}$$

with a Lindblad generator $i[h_0, \cdot] + \Gamma_\beta(\cdot)$ explicited in (5.17)

Remarks

(i) Let us make a comparison of the above with the results of Ref. 3, Sec. 4.2, which concern similar generators as ours. More precisely, (2.6) corresponds to a particular case of the Hamiltonian of Eq. (15) in Ref. 3 with $D_{ij} = 0, \forall i, j$. In Ref. 3 the choice of time scale τ and coupling λ is such that $\lambda^2\tau = 1$, $\tau \to 0$. A supplementary structure is present in that work which consists in making the suitably renormalized spins forming the chain merge in the limit $\tau \to 0$ to yield a heat bath represented by a Fock space of quantum noises. The limit $\tau \to 0$ performed in the language adopted in Ref. 3 exists and yields a quantum Langevin equation for the whole limiting system consisting in the original small system in interaction with a field of quantum noises. When restricted to $\mathcal{H}_0$, the effective dynamics of observables at zero temperature corresponds to a contraction semigroup generated by

$$\Gamma_\infty(\cdot) = i[h_0, \cdot] + \sum_{m=1}^{n} \left(V_m^* \cdot V_m - \frac{1}{2}(V_m^* V_m \cdot + \cdot V_m^* V_m) \right), \qquad (5.21)$$

which coincides with Theorem 5.3 at $\beta = \infty$.

(ii) The generator Γ^β coincides with the generator (4.62) obtained in Corollary 4.1 in the scaling $\lambda^2\tau \to 0$, $\tau \to 0$, modulo the # operation, which appears as a trade mark of the perturbative regime.

5.3. The Continuous Limit

For completeness, we mention here the easier cases of continuous limit characterized by $\tau \to 0$ and λ constant. The omitted proof are quite analogous to those of the previous Section.

First considering the Schrödinger picture, we get

Proposition 5.1. *Assume the hypothesis* ***H3*** *holds and fix* $\lambda = 1$. *Then,*

$$s - \lim_{\substack{\tau\to 0\\ t/\tau\in\mathbb{N}}} PU(t/\tau, 0)P = s - \lim_{\substack{\tau\to 0\\ t/\tau\in\mathbb{N}}} \left[Pe^{-i\tau(H(0)+W)}P\right]^{t/\tau} = e^{-itH(0)_P}. \qquad (5.22)$$

The Heisenberg evolution also yields a unitary effective evolution in the continuous limit:

Proposition 5.2. *Consider the matrix model of Sec. 2 and fix* $\lambda = 1$. *Then,*

$$\lim_{\substack{\tau\to 0\\ t/\tau\in\mathbb{N}}} B_\beta(t/\tau, \tau, 1) = e^{i[h_0,\cdot]}(B). \qquad (5.23)$$

In order to make explicit the results of Sec. 4, we provide below a detailed analysis of the case $d = n = 1$.

6. THE CASE $d = n = 1$

In that Section, we focus on the first non-trivial case where the small system lives on $\mathbb{C}^2$ and the heat bath is formed by a chain of spins 1/2. We provide explicit formulas for T_β and $T_\beta^\#$ which are valid for any coupling operator V appearing in (2.8). We further diagonalize the restriction of T_β to the degenerate subspace $Ker(\mathcal{U}_{0,0}(0) - 1)$ in order to determine the subalgebra of observables invariant under the effective dynamics in the weak coupling limit (keeping τ fixed).

For $\mathcal{H}_0 = \mathbb{C}^2$, $\mathcal{H} = \otimes_{j\geq 1}\mathbb{C}^2$, we write for $t \in [\tau(k-1), \tau k[$ in $\mathcal{H}_0 \otimes C_k^2$,

$$H(t, \lambda) = H(\lambda) = H(0) + \lambda W, \tag{6.1}$$

where

$$H(0) = h_0 \otimes \mathbb{I} + \mathbb{I} \otimes \delta a^* a, \quad W = V^* \otimes a + V \otimes a^*. \tag{6.2}$$

We choose, without loss of generality, $h_0 = \epsilon\sigma_z$, $\epsilon \neq 0$, so that we have in the ordered basis $\{\omega \otimes \omega, x \otimes \omega, \omega \otimes x, x \otimes x\}$

$$H(\lambda) = \begin{pmatrix} \epsilon\sigma_z & \lambda V^* \\ \lambda V & \delta\mathbb{I} + \epsilon\sigma_z \end{pmatrix}, \quad \text{with the convention} \quad \sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.3}$$

Specifying the results of the previous sections to the case under study, we can write, uniformly in β, as $\lambda \to 0$,

$$\mathcal{U}_\beta(\lambda) = \mathcal{U}_{0,0}(0) + \lambda^2 \frac{T_\beta}{1 + e^{-\delta\beta}} + O(\lambda^4), \tag{6.4}$$

with

$$\begin{aligned} T_\beta(B) = {} & F_{0,1}(-\tau) B F_{1,0}(\tau) + G_{0,0}(-\tau) B e^{-i\tau H_{0,0}(0)} + e^{i\tau H_{0,0}(0)} B G_{0,0}(\tau) \\ & + e^{-\delta\beta}\big(F_{1,0}(-\tau) B F_{0,1}(\tau) + G_{1,1}(-\tau) B e^{-i\tau H_{1,1}(0)} + e^{i\tau H_{1,1}(0)} B G_{1,1}(\tau)\big). \end{aligned} \tag{6.5}$$

We use the norm induced by the scalar product $\langle A, B\rangle = \text{Tr}(A^* B)$, i.e. the Hilbert-Schmidt norm. As easily verified, an orthonormal basis of eigenvectors for the unitary operator $\mathcal{U}_{0,0}(0)(\cdot) = e^{i\tau\epsilon\sigma_z} \cdot e^{-i\tau\epsilon\sigma_z}$, with associated eigenvalues, is provided by

$$\{-\hat{\mathbb{I}}, \hat{\sigma}_z, \sigma_-, \sigma_+\} \longleftrightarrow \{1, 1, e^{-2i\tau\epsilon}, e^{2i\tau\epsilon}\}, \tag{6.6}$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\mathbb{I}} = \mathbb{I}/\sqrt{2} \quad \text{and} \quad \hat{\sigma}_z = \sigma_z/\sqrt{2}. \tag{6.7}$$

Let us compute T_β restricted to the subspace Ker $(\mathcal{U}_{0,0}(0) - 1)$ appearing in $T_\beta^\#$.

Lemma 6.1. *With respect to the orthonormal basis $\{\hat{\mathbb{I}}, \hat{\sigma}_z\}$, and with the notation*

$$A^{OD} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \quad \text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \tag{6.8}$$

we have

$$T_\beta|_{\{\hat{\mathbb{I}}, \hat{\sigma}_z\}} = \begin{pmatrix} 0 & T_{\beta_{1,2}} \\ 0 & T_{\beta_{2,2}} \end{pmatrix} \tag{6.9}$$

where

$$\begin{aligned} T_{\beta_{1,2}} &= \left(\left|\left(F^{OD}_{1,0}\right)_{2,1}\right|^2 - \left|\left(F^{OD}_{1,0}\right)_{1,2}\right|^2\right)(1 - e^{-\delta\beta}) \\ T_{\beta_{2,2}} &= -\left(\left\|F^{OD}_{1,0}\right\|^2 + e^{-\delta\beta}\left\|F^{OD}_{0,1}\right\|^2\right) = -\left\|F^{OD}_{1,0}\right\|^2(1 + e^{-\delta\beta}) \leq 0. \end{aligned} \tag{6.10}$$

Furthermore, if in (2.8) $V = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$,

$$F_{1,0}(\tau) = -i \begin{pmatrix} e^{i\tau(\epsilon-\delta)} \int_0^\tau e^{is\delta} ds\, \mathbf{a} & e^{i\tau(\epsilon-\delta)} \int_0^\tau e^{is(\delta-2\epsilon)} ds\, \mathbf{b} \\ e^{-i\tau(\epsilon+\delta)} \int_0^\tau e^{is(\delta+2\epsilon)} ds\, \mathbf{c} & e^{-i\tau(\epsilon+\delta)} \int_0^\tau e^{is\delta} ds\, \mathbf{d} \end{pmatrix} \tag{6.11}$$

Proof The first column is proportional to $T_\beta(\mathbb{I}) = 0$. The second column of the matrix is given by

$$\frac{1}{2}\begin{pmatrix} \mathrm{Tr}(T_\beta(\sigma_z)) \\ \mathrm{Tr}(\sigma_z T_\beta(\sigma_z)) \end{pmatrix}. \tag{6.12}$$

where, dropping the positive argument τ in F and further making use of (3.36) and (3.37),

$$\begin{aligned} T_\beta(\sigma_z)) = {} & F^*_{1,0}\sigma_z F_{1,0} + G^*_{0,0}\sigma_z e^{-i\tau H_{0,0}(0)} + e^{i\tau H_{0,0}(0)}\sigma_z G_{0,0} \\ & + e^{-\delta\beta}\left(F^*_{0,1}\sigma_z F_{0,1} + G^*_{1,1}\sigma_z e^{-i\tau H_{1,1}(0)} + e^{i\tau H_{1,1}(0)}\sigma_z G_{1,1}\right). \end{aligned} \tag{6.13}$$

Further making use of the cyclicity of the trace, $[\sigma_z, H_{n,n}(0)] = 0$, $\sigma_z^2 = \mathbb{I}$ and of (3.36) and (3.37) again, we can write

$$\begin{aligned} \mathrm{Tr}(\sigma_z T_\beta(\sigma_z))) = {} & \mathrm{Tr}(\sigma_z F^*_{1,0}\sigma_z F_{1,0}) - \mathrm{Tr}(F^*_{1,0}F_{1,0}) \\ & + e^{-\delta\beta}(\mathrm{Tr}(\sigma_z F^*_{0,1}\sigma_z F_{0,1}) - \mathrm{Tr}(F^*_{0,1}F_{0,1})). \end{aligned} \tag{6.14}$$

Explicit computations on 2×2 matrices yields the first equality in (6.10). Let us turn to (6.11). From the definitions (3.23) and (2.8), we have

$$F(\tau) = \begin{pmatrix} \mathbb{O} & F_{0,1}(\tau) \\ F_{1,0}(\tau) & \mathbb{O} \end{pmatrix}, \tag{6.15}$$

where

$$F_{0,1}(\tau) = -i\int_0^\tau e^{-i(\tau-s)H_{0,0}(0)}V^* e^{-isH_{1,1}(0)}ds, \tag{6.16}$$

$$F_{0,1}(\tau) = -i\int_0^\tau e^{-i(\tau-s)H_{0,0}(0)}V e^{-isH_{0,0}(0)}ds. \tag{6.17}$$

By explicit computations with V as in the statement, we obtain

$$F_{0,1}(\tau) = -i\begin{pmatrix} e^{i\tau\epsilon}\int_0^\tau e^{-is\delta}ds\ \bar{\mathbf{a}} & e^{i\tau\epsilon}\int_0^\tau e^{-is(\delta+2\epsilon)}ds\ \bar{\mathbf{c}} \\ e^{-i\tau\epsilon}\int_0^\tau e^{-is(\delta-2\epsilon)}ds\ \bar{\mathbf{b}} & e^{-i\tau\epsilon}\int_0^\tau e^{-is\delta}ds\ \bar{\mathbf{d}} \end{pmatrix} \tag{6.18}$$

$$F_{1,0}(\tau) = -i\begin{pmatrix} e^{i\tau(\epsilon-\delta)}\int_0^\tau e^{is\delta}ds\ \mathbf{a} & e^{i\tau(\epsilon-\delta)}\int_0^\tau e^{is(\delta-2\epsilon)}ds\ \mathbf{b} \\ e^{-i\tau(\epsilon+\delta)}\int_0^\tau e^{is(\delta+2\epsilon)}ds\ \mathbf{c} & e^{-i\tau(\epsilon+\delta)}\int_0^\tau e^{is\delta}ds\ \mathbf{d} \end{pmatrix}, \tag{6.19}$$

which yields the expression for $T_{\beta_{i,j}}$.
By similar manipulations we get

$$\begin{aligned}\mathrm{Tr}(\mathbb{I}T_\beta(\sigma_z)) &= \mathrm{Tr}(F_{1,0}^*\sigma_z F_{1,0}) - \mathrm{Tr}(\sigma_z F_{1,0}^* F_{1,0}) \\ &\quad + e^{-\delta\beta}(\mathrm{Tr}(F_{0,1}^*\sigma_z F_{0,1}) - \mathrm{Tr}(\sigma_z F_{0,1}^* F_{0,1})).\end{aligned} \tag{6.20}$$

Now, for any $F \in M_2(\mathbb{C})$,

$$\mathrm{Tr}(\sigma_z(FF^* - F^*F)) = 2(|F_{21}|^2 - |F_{12}|^2), \tag{6.21}$$

so that we get the first line of (6.10). □

We also need to compute $\mathrm{Tr}(\sigma_- T_\beta(\sigma_+))$ and $\mathrm{Tr}(\sigma_+ T_\beta(\sigma_-))$ to get $T_\beta^\#$.

Lemma 6.2. *By explicit computation and Lemma 4.7, we have*

$$\begin{aligned}Tr(\sigma_- T_\beta(\sigma_+)) &= \overline{Tr(\sigma_+ T_\beta(\sigma_-))} \\ &= (F_{1,0})_{1,1}\overline{(F_{1,0})_{2,2}} + e^{i\tau\epsilon}((G_{0,0})_{1,1} + \overline{(G_{0,0})_{2,2}}) \\ &\quad + e^{-\delta\beta}\left((F_{0,1})_{1,1}\overline{(F_{0,1})_{2,2}} + e^{i\tau\epsilon}(e^{i\tau\delta}(G_{1,1})_{1,1} + e^{-i\tau\delta}\overline{(G_{1,1})_{2,2}})\right).\end{aligned} \tag{6.22}$$

It remains to diagonalize the restriction of T_β to $\mathrm{span}(\hat{\mathbb{I}}, \hat{\sigma}_z)$ to have a complete description of the generator of the effective evolution. Introducing

$$\mu = T_{\beta_{1,2}}, \quad \nu = T_{\beta_{2,2}}, \tag{6.23}$$

we actually get by perturbation theory,

Lemma 6.3. *Assume $\epsilon\tau \notin \mathbb{Z}\pi$. Then, for $\lambda > 0$, there exists a continuous set of eigenprojectors and eigenvalues of $\mathcal{U}_\beta(\lambda, \tau)$ denoted respectively by $\{\Pi_j(\lambda)\}_{j=1,\ldots,4}$ and $\{u_j(\lambda)\}_{j=1,\ldots,4}$ such that*

$$
\begin{aligned}
u_1(\lambda) &= 1 + O(\lambda^4),\\
u_2(\lambda) &= 1 - \lambda^2 \left\| F_{1,0}^{OD} \right\|^2 + O(\lambda^4), \qquad (6.24)\\
u_3(\lambda) &= e^{2i\tau\epsilon} + \lambda^2 Tr(\sigma_- T_\beta(\sigma_+)) + O(\lambda^4),\\
u_4(\lambda) &= e^{-2i\tau\epsilon} + \lambda^2 Tr(\sigma_- + T_\beta(\sigma_-)) + O(\lambda^4),
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_1(\lambda)(B) &= \frac{Tr((\mathbb{I} - \frac{\mu}{\nu}\sigma_z)B)}{2}\mathbb{I} + O(\lambda^2),\\
\Pi_2(\lambda)(B) &= \frac{Tr(\sigma_z B)}{2}\left(\frac{\mu}{\nu}\mathbb{I} + \sigma_z\right) + O(\lambda^2)\\
\Pi_3(\lambda)(B) &= Tr(\sigma_- B)\sigma_+ + O(\lambda^2),\\
\Pi_4(\lambda)(B) &= Tr(\sigma_+ B)\sigma_- + O(\lambda^2). \qquad (6.25)
\end{aligned}
$$

Moreover, $\Pi_0 := \Pi_1(0) + \Pi_2(0)$, $\Pi_3(0)$ *and* $\Pi_4(0)$ *are the spectral projectors of* $\mathcal{U}_{0,0}(0)$ *and* $\{\Pi_j(0)\}_{j=1,\ldots,4}$ *are those of* T_β.

Hence, we obtain the

Proposition 6.1. *Let $t/\lambda^2 = k \in \mathbb{N}$, and consider the Hamiltonian (6.3). Then*

$$
\lim_{\lambda\to 0} \mathcal{U}_{0,0}(0)^{-t/\lambda^2} B(t/\lambda^2, \beta, \lambda) = \lim_{\lambda\to 0} \mathcal{U}_{0,0}(0)^{-t/\lambda^2} \mathcal{U}_\beta(\lambda, \tau)^{t/\lambda^2}(B) = e^{t\Gamma_\beta^w}(B), \qquad (6.26)
$$

were

$$
\begin{aligned}
\Gamma_\beta^w = \frac{1}{1+e^{-\delta\beta}}\big(&- \left\| F_{1,0}^{OD} \right\| \Pi_2(0)\\
&+ e^{-2i\tau\epsilon} Tr(\sigma_- T_\beta(\sigma_+)\big)\Pi_3(0) + e^{2i\tau\epsilon} Tr(\sigma_+ T_\beta(\sigma_-))\Pi_4(0)). \qquad (6.27)
\end{aligned}
$$

The dynamics of any observable is thus fully determined from these formulas.

ACKNOWLEDGMENTS

We wish to thank Laurent Bruneau for a careful and critical reading of the manuscript and Claude-Alain Pillet for useful discussions.

REFERENCES

1. R. Alicki and M. Fannes, *Quantum Dynamical Systems*, Oxford University Press, (2001).
2. S. Attal and M. Lindsay (Eds), *Quantum Probability Communictions XI and XII*, World Scientific, (2003).
3. S. Attal and Y. Pautrat, From Repeated to Continuous Quantum Interactions, *Annales Henri Poincaré*, **7**:59–104 (2006).
4. A. Buchleitner and K. Hornberger (Eds), *Coherent Evolution in Noisy Environments*, Springer Lecture Notes in Physics 611, (2002).
5. O. Brattelli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, *Texts and Monographs in Physics*, Springer, New York, Heidelberg, Berlin, (1981).
6. E. B. Davies, Markovian Master Equations, *Comm. Math. Phys.* **39**:91–110 (1974).
7. E. B. Davies, *One-Parameter Semigroups*, Academic Press, (1980).
8. E. B. Davies, *Quantum Theory of Open Systems*, Academic Press, (1976).
9. J. Derezinski and V. Jaksic, On the Nature of Fermi Golden Rule of Open Quantum Systems, *J. Stat. Phys.* **116**: 411–423 (2004).
10. E. B. Davies and H. Spohn, Open Quantum Systems with Time-Dependent Hamiltonians and their Linear Response *J. Stat. Phys.* **19**:511–523 (1978).
11. T. Kato, *Perturbation Theory for Linear Operators*, Springer, (1980).
12. J. Lebowitz and H. Spohn, Irreversible Thermodynamics for Quantum Systems Weakly Coupled to Thermal Reservoirs, *Adv. Chem. Phys.* **39**:109–142 (1978).
13. J. M. Lindsay and H. Maassen, Stochastic Calculus for Quantum Brownian Motion of a non-minimal variance. In *Mark Kac Seminar of probability in Physics*, Syllabus 1987–1992. CWI Syllabus 32, Amsterdam (1992).
14. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, (1983).
15. T. Wellens, A. Buchleitner, B. Kuemmerer, and H. Maassen, Quantum State Preparation via Asymptotic Completeness, *Phys. Rev. Lett.* **85**:3361 (2000).